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***Research Article***

# MetaStructure, Meta-HyperStructure, and Meta-SuperHyper Structure

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## Article Info

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## Abstract

This paper investigates new hierarchical frameworks in mathematics by extending the notion of classical structures. A *Structure* refers to an arbitrary mathematical framework, including but not limited to those from Set Theory, Logic, Probability, Statistics, Algebra, and Geometry. A *HyperStructure* generalizes classical algebraic structures by replacing the underlying set $S$ with its powerset $\mathscr{P}(S)$, where hyperoperations combine subsets into (possibly new) subsets, enabling the expression of higher–order relations. A *SuperHyperStructure* further extends this idea by recursively applying the powerset construction, capturing increasingly complex hierarchical interactions.

In addition, we introduce and study the concepts of *MetaStructure*, *Iterated MetaStructure*, *Meta-HyperStructure*, and *Meta-SuperHyperStructure*. A MetaStructure treats entire mathematical structures as objects, with meta-operations producing new structures. An Iterated MetaStructure arises by repeatedly applying the MetaStructure construction, thereby forming hierarchical layers where structures of structures yield progressively deeper meta-level frameworks. These approaches allow various mathematical and scientific concepts to be unified and generalized from a meta-level perspective, offering a systematic foundation for exploring higher-order structures across diverse fields.

## 1. Preliminaries

This section presents the fundamental concepts and definitions that underpin the discussions in this paper. All concepts considered in this paper are assumed to be finite.

### 1.1. Classical Structure

In this paper, the term *Structure* refers to an arbitrary mathematical structure, including but not limited to those from the domains of Set Theory, Logic, Probability, Statistics, Algebra, and Geometry.

**Definition 1.1** (Classical Structure)**.** *[1] A* Classical Structure $\mathscr{C}$ *is a mathematical structure drawn from one of various domains—such as Set theory, Logic, Probability, Statistics, Algebra, Geometry, Graph theory, Automata theory, Game theory, etc.—and can be formalized as a pair*

$$\mathscr{C} = \big(H, \{\#^{(m)}\}_{m\in\mathscr{I}}\big),$$

*where:*

- *$H$ is a nonempty set (the* carrier *or* universe*).*

- *For each $m \in \mathscr{I} \subseteq \mathbb{Z}_{>0}$, there is an m-ary operation*

$$\#^{(m)} : H^m \longrightarrow H,$$

*subject to specified* axioms *(e.g., associativity, commutativity, identity laws) depending on the particular type of structure.*

*We say that $\mathscr{C}$ is of* type $\{\#^{(m)} : m \in \mathscr{I}\}$*. Examples include:*

- *A* Set $(S, \varnothing)$ *viewed as a carrier with distinguished elements or relations, but no additional operations[2].*
- *A* Logic $(L, \wedge, \vee, \neg)$*, where $\wedge$ and $\vee$ are binary connectives and $\neg$ is a unary connective satisfying logical axioms.*
- *A* Probability *structure $(\Omega, \mathscr{F}, P)$, where $P : \mathscr{F} \to [0,1]$ is a measure on a sigma-algebra $\mathscr{F} \subseteq \mathscr{P}(\Omega)$ [3, 4].*
- *A* Statistics *model $(X, \mathscr{A}, \theta)$, where $\theta$ maps data $X$ to statistical parameters[5, 6].*
- *A* Algebraic *structure:*
  - *A* Group $(G, *)$ *with $* : G \times G \to G$ satisfying associativity, identity, and inverses[7].*
  - *A* Ring $(R, +, \times)$ *with two binary operations satisfying ring axioms[8].*
  - *A* Vector Space $(V, +, \cdot)$ *over a field $\mathbb{F}$, where $\cdot : \mathbb{F} \times V \to V$[9, 10].*
- *A* Geometric *structure* $(X, \mathrm{dist})$*, where* $\mathrm{dist} : X \times X \to \mathbb{R}$ *satisfies the metric axioms.*
- *A* Graph $(V, E)$*, where $E \subseteq \{\{u, v\} \mid u, v \in V\}$ in the undirected case (or $E \subseteq V \times V$ in the directed case), with additional notions of adjacency and incidence[11–13].*
- *An* Automaton $(Q, \Sigma, \delta, q_0, F)$*, where $Q$ is a finite set of states, $\Sigma$ is an input alphabet, $\delta : Q \times \Sigma \to Q$ is the transition function, $q_0 \in Q$ is the start state, and $F \subseteq Q$ is the set of accepting states[14, 15].*
- *A* Game $(N, \{A_i\}, \{u_i\})$*, where $N$ is a set of players, $A_i$ is the action set for player i, and $u_i : \prod_{j \in N} A_j \to \mathbb{R}$ is the utility (payoff) function for player i.*

## 1.2. Hyperstructure and SuperHyperStructure

Many mathematical and real-world concepts exhibit inherently hierarchical structures. To capture and represent such multi-level relationships in a clear and systematic way, the notions of *hyperstructure* and *superhyperstructure* have been introduced. A *hyperstructure* generalizes a classical algebraic structure by replacing the underlying set $S$ with its powerset $\mathscr{P}(S)$; hyperoperations then combine subsets into (possibly new) subsets, enabling the expression of higher–order relations [16–20]. Related concepts include HyperFuzzy Sets [21–23], HyperAlgebras [24, 25], Chemical HyperStructure [26–30], and HyperGraphs [31–34].

**Definition 1.2** (Base set)**.** *[35] Let $S$ be a nonempty set, called the* base set*. Equivalently,*

$$S = \{\, x \mid x \text{ is an element of the underlying universe} \,\}.$$

*All further constructions—such as $\mathscr{P}(S)$ or its iterates $\mathscr{P}_n(S)$—are formed from elements of $S$.*

**Definition 1.3** (Powerset)**.** *[2] The* powerset *of a set $S$, denoted $\mathscr{P}(S)$, is the collection of all subsets of $S$:*

$$\mathscr{P}(S) = \{\, A \mid A \subseteq S \,\},$$

*including both the empty set $\emptyset$ and $S$ itself.*

**Definition 1.4** (Hyperoperation)**.** *(cf. [18, 36, 37])* A hyperoperation *is a generalization of a binary operation in which the result of combining two inputs is a* set *(not necessarily a singleton). Formally, for a set $S$, a hyperoperation $\circ$ is a map*

$$\circ : S \times S \longrightarrow \mathscr{P}(S).$$

**Definition 1.5** (Hyperstructure)**.** (cf. [38–42]) *A* hyperstructure *replaces the base set $S$ by its powerset and is given by*

$$\mathscr{H} = \big(\mathscr{P}(S), \circ\big),$$

*where $\circ$ is a hyperoperation acting on subsets of $S$ (i.e., $\circ : \mathscr{P}(S) \times \mathscr{P}(S) \to \mathscr{P}(S)$). This framework allows one to combine collections of elements into other collections.*

**Example 1.6** (Retail co–purchase expansion (recommendation closure))**.** *Let $S$ be the set of all products in an online shop. From historical baskets, mine a directed co–purchase relation $R \subseteq S \times S$ where $(x, p) \in R$ means "customers who buy x often also buy p." For $X \subseteq S$ define the (deterministic) closure*

$$\mathrm{Cl}_R(X) \;:=\; X \,\cup\, \{\, p \in S \mid \exists x \in X : \; (x, p) \in R \,\}.$$

*Define the hyperoperation on subsets*

$$\circ \;:\; \mathscr{P}(S) \times \mathscr{P}(S) \longrightarrow \mathscr{P}(S), \qquad A \circ B \;:=\; \mathrm{Cl}_R(A \cup B).$$

*Then $\mathscr{H} = \big(\mathscr{P}(S), \circ\big)$ is a hyperstructure: inputs are* collections *of products, and the output is the expanded collection produced by combining and closing under co–purchase. The operator $\mathrm{Cl}_R$ is extensive, monotone, and idempotent, hence $\circ$ inherits natural stability properties.*

***Toy computation.*** *Let*

$$S = \{\mathsf{laptop}, \mathsf{mouse}, \mathsf{bag}, \mathsf{coffee}, \mathsf{mug}\}$$

*and*

$$R = \{(\mathit{laptop}, \mathit{mouse}), (\mathit{laptop}, \mathit{bag}), (\mathit{coffee}, \mathit{mug})\}.$$

*For* $A = \{\mathit{laptop}\}$ *and* $B = \{\mathit{coffee}\}$*,*

$$A \circ B = \mathrm{Cl}_R(\{\mathit{laptop}, \mathit{coffee}\})$$

$$= \{\mathit{laptop}, \mathit{coffee}, \mathit{mouse}, \mathit{bag}, \mathit{mug}\}.$$

*Thus the hyperoperation bundles two customer segments and deterministically yields the combined, recommendation–expanded shelf.*

**Example 1.7** (Transit coverage with one transfer (two–hop neighborhood))**.** *Let S be the set of city blocks (or stops) and let* $G = (S, E)$ *be the undirected adjacency graph encoding walkability or same–line connections. For* $X \subseteq S$*, define the radius–r neighborhood*

$$N_r(X) := \{y \in S \mid \exists x \in X \text{ with graph distance } d_G(x, y) \le r\}.$$

*Model two services (e.g. bus and bike–share) by* $A, B \subseteq S$ *(their initial coverage). Define the hyperoperation*

$$\circ : \mathscr{P}(S) \times \mathscr{P}(S) \to \mathscr{P}(S), \qquad A \circ B := N_2(A \cup B),$$

*i.e., "combined coverage after allowing a single transfer" (two hops). Then* $\mathscr{H} = (\mathscr{P}(S), \circ)$ *is a hyperstructure that maps two coverage regions to the deterministically induced two–hop service area. Note that* $\circ$ *is monotone and idempotent (*$A \circ A = N_2(A)$*, and* $N_2(N_2(A)) = N_2(A)$*).*

***Toy computation.*** *Let* $S = \{1, 2, 3, 4, 5, 6\}$ *with edges* $E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 6\}\}$ *(a path). For* $A = \{2\}$ *(bus) and* $B = \{5\}$ *(bike–share),*

$$A \circ B = N_2(\{2, 5\}) = \{1, 2, 3, 4, 5, 6\},$$

*so allowing at most one transfer from either service covers the whole corridor.*

A *SuperHyperStructure* extends the concept of a hyperstructure by recursively applying the powerset construction $n$ times. In this framework, operations act on nested collections, thereby enabling the modeling of hierarchical, multi-level interactions [18, 20, 43–46]. Related notions in the literature include the *SuperHyperGraph* [47–49] and the *SuperHyperUncertain Set* [50–53].

**Definition 1.8** (Iterated powersets)**.** *(cf. [25, 38, 43, 54]) For* $n \ge 1$*, define recursively*

$$\mathscr{P}_1(S) = \mathscr{P}(S), \qquad \mathscr{P}_{n+1}(S) = \mathscr{P}\big(\mathscr{P}_n(S)\big).$$

*Similarly, the* nonempty *iterated powersets are*

$$\mathscr{P}_1^*(S) = \mathscr{P}(S) \setminus \{\emptyset\}, \qquad \mathscr{P}_{n+1}^*(S) = \mathscr{P}^*\big(\mathscr{P}_n^*(S)\big),$$

*where* $\mathscr{P}^*(X) = \mathscr{P}(X) \setminus \{\emptyset\}$*.*

**Example 1.9** (Iterated powersets in a photo–album app (real–life model))**.** *Fix a finite base set of digital photos*

$$S = \{p_1, p_2, p_3, p_4\}.$$

*In a typical photo application:*

- *Elements of the* powerset $\mathscr{P}(S)$ *are* ***albums*** *(each album is a subset of photos).*
- *Elements of the* iterated powerset $\mathscr{P}_2(S) = \mathscr{P}(\mathscr{P}(S))$ *are* ***curated groups of albums*** *(e.g., "Summer Trips").*
- *Elements of* $\mathscr{P}_3(S) = \mathscr{P}(\mathscr{P}_2(S))$ *are* ***collections of curated groups*** *(e.g., a publisher's multi–curator bundle).*

***Concrete instance.*** *Define albums (elements of* $\mathscr{P}(S)$*)*

$$A = \{p_1, p_4\}, \qquad B = \{p_2\}, \qquad C = \{p_2, p_3\}.$$

*Define curated groups (elements of* $\mathscr{P}_2(S)$*)*

$$G = \{A, C\}, \qquad H = \{B, C\}.$$

*Finally, define a collection of curated groups (an element of* $\mathscr{P}_3(S)$*)*

$$\mathscr{C} = \{G, H\}.$$

***Flattening ("view as photos").*** *There are natural* flattening *maps (unions):*

$$\bigcup : \mathscr{P}_2(S) \longrightarrow \mathscr{P}(S), \quad \bigcup \mathfrak{G} := \bigcup_{A \in \mathfrak{G}} A, \qquad \bigcup : \mathscr{P}_3(S) \longrightarrow \mathscr{P}_2(S), \quad \bigcup \mathscr{C} := \bigcup_{\mathfrak{G} \in \mathscr{C}} \mathfrak{G}.$$

*For our data,*

$$\bigcup G = A \cup C = \{p_1, p_2, p_3, p_4\}, \qquad \bigcup H = B \cup C = \{p_2, p_3\},$$

$$\bigcup \mathscr{C} = G \cup H = \{A, B, C\} \in \mathscr{P}_2(S), \qquad \bigcup\big(\bigcup \mathscr{C}\big) = A \cup B \cup C = \{p_1, p_2, p_3, p_4\}.$$

***Operations at different levels.*** *Set operations act at their respective levels and need not "collapse" types:*

$$G \cap H = \{C\} \in \mathscr{P}_2(S), \qquad \bigcup(G \cap H) = \bigcup\{C\} = C = \{p_2, p_3\}.$$

*By contrast,*

$$(\bigcup G) \cap (\bigcup H) = \{p_1, p_2, p_3, p_4\} \cap \{p_2, p_3\} = \{p_2, p_3\},$$

*which matches $\bigcup(G \cap H)$ but lives in $\mathscr{P}(S)$ (photos level), not in $\mathscr{P}_2(S)$ (albums level).*
***Interpretation.***

- *S: individual photos.*
- *$\mathscr{P}(S)$: albums (each album is a subset of photos).*
- *$\mathscr{P}_2(S)$: curated groups of albums (e.g., by theme, trip, or project).*
- *$\mathscr{P}_3(S)$: higher–level bundles (e.g., publisher packs combining multiple curators' groups).*

*This faithfully models a real workflow where users manipulate photos, albums, and collections–of–albums. Iterated powersets capture these layers without losing the distinction between "photo–level" and "album–level" operations.*

**Definition 1.10** (SuperHyperOperation)**.** *[38] Let H be a nonempty set. Define recursively, for each integer $k \geq 0$,*

$$\mathscr{P}^0(H) = H, \qquad \mathscr{P}^{k+1}(H) = \mathscr{P}\big(\mathscr{P}^k(H)\big).$$

*Fix $m, n \geq 0$. An $(m,n)$-*SuperHyperOperation *is a map*

$$\circ^{(m,n)} : H^m \longrightarrow \mathscr{P}^n_*(H),$$

*where $\mathscr{P}^n_*(H)$ denotes the n-th iterated powerset of H, either excluding the empty set (classical type) or including it (Neutrosophic type). In the former case we call $\circ^{(m,n)}$ a* classical-type $(m,n)$-SuperHyperOperation*; in the latter, a* Neutrosophic $(m,n)$-SuperHyperOperation*.*

**Definition 1.11** (SuperHyperStructure of order $(m,n)$)**.** *(cf. [18, 55–57]) Let S be a nonempty set and $m, n \geq 0$. A $(m,n)$-*SuperHyperStructure *on S of arity k is a map*

$$\star : \underbrace{\mathscr{P}^m(S) \times \cdots \times \mathscr{P}^m(S)}_{k \text{ factors}} \longrightarrow \mathscr{P}^n(S).$$

*When $m = 0$ and $n = 0$, this recovers an ordinary k-ary operation on S; when $m = 0$, $n = 1$, it is a k-ary hyperoperation; and when $k = 1$, it is an $(m,n)$-superhyperfunction.*

**Example 1.12** ((m,n)=(1,2), $k = 2$ — Music streaming: playlists and "collections of playlists")**.** *Let S be the set of all tracks in a streaming app. Elements of $\mathscr{P}(S)$ are* playlists *(subsets of tracks), while elements of $\mathscr{P}^2(S)$ are* collections of playlists *(curated bundles).*
*Fix the arity $k = 2$ and define*

$$\star : \ \mathscr{P}(S) \times \mathscr{P}(S) \longrightarrow \mathscr{P}^2(S)$$

*by the rule*

$$\star(G_1, G_2) \ := \ \Big\{ \ P \subseteq S \ \Big| \ |P| \leq L, \ P \cap G_1 \neq \varnothing, \ P \cap G_2 \neq \varnothing \ \Big\},$$

*where $L \in \mathbb{N}$ is a small curation cap (e.g. $L = 3$). Thus $\star(G_1, G_2)$ is a* set of playlists*, i.e. an element of $\mathscr{P}^2(S)$.*

***Concrete instance.*** *Let $S = \{t_1, t_2, t_3, t_4\}$, $L = 3$, $G_1 = \{t_1, t_2\}$ ("upbeat"), and $G_2 = \{t_3\}$ ("acoustic"). Then*

$$\star(G_1, G_2) = \{ \ \{t_1, t_3\}, \ \{t_2, t_3\}, \ \{t_1, t_2, t_3\}, \ \{t_1, t_3, t_4\}, \ \{t_2, t_3, t_4\} \ \} \ \in \ \mathscr{P}^2(S).$$

*Interpretation: from two seed themes $G_1, G_2$, the app proposes a* collection of candidate playlists *that mix (at least) one track from each theme, never exceeding the editorial cap L.*

**Example 1.13** ((m,n)=(0,1), $k = 2$ — Grocery recommender: pair $\rightarrow$ suggested complements)**.** *Let S be a set of ingredients in a grocery or recipe app. Elements of $\mathscr{P}(S)$ are* suggested complements*. Fix $k = 2$ and define a hyperoperation*

$$\star : \ S \times S \longrightarrow \mathscr{P}(S)$$

*as follows. For each $x \in S$ let* $\mathrm{Comp}(x) \subseteq S$ *be a (domain–curated) set of complements. Set*

$$\star(x, y) \ := \ \big(\mathrm{Comp}(x) \cup \mathrm{Comp}(y)\big) \setminus \{x, y\}.$$

*Hence $(m,n) = (0,1)$: two* items *in, a* set *of complementary items out.*

***Concrete instance.*** *Take*

$$S = \{bread, \ cheese, \ ham, \ tomato, \ pasta, \ basil, \ olive\ oil\},$$

*with*

$$\mathrm{Comp}(bread) = \{cheese, ham\}, \qquad \mathrm{Comp}(cheese) = \{bread, tomato\},$$
$$\mathrm{Comp}(pasta) = \{tomato, olive\ oil, basil\}, \qquad \mathrm{Comp}(tomato) = \{basil\}.$$

*Then*

$$\star(bread, cheese) = \big(\{cheese, ham\} \cup \{bread, tomato\}\big) \setminus \{bread, cheese\} = \{ham, tomato\}.$$

*Interpretation: given two items in the basket, the system proposes a* set *of add–ons that pair well with at least one of them.*

**Example 1.14** ((m,n)=(2,0), $k = 1$ — Federation of committees → elected representative)**.** *Let $S$ be a set of people in an organization. Elements of $\mathscr{P}(S)$ are* committees*; elements of $\mathscr{P}^2(S)$ are* federations of committees *(collections of committees). Define a superhyperfunction ($k = 1$)*

$$\star : \mathscr{P}^2(S) \longrightarrow S$$

*that selects a single representative from a federation $F \in \mathscr{P}^2(S)$ by a simple majority–presence rule:*

$$\mathrm{score}_F(x) := \#\{C \in F \mid x \in C\}, \qquad \star(F) := \arg\max_{x \in \bigcup F} \mathrm{score}_F(x),$$

*with a fixed, deterministic tie–breaker (e.g. alphabetical order).*

***Concrete instance.*** *Let $S = \{a,b,c,d\}$ and committees*

$$C_1 = \{a,b\}, \quad C_2 = \{b,c\}, \quad C_3 = \{b,d\}, \qquad F = \{C_1,C_2,C_3\} \in \mathscr{P}^2(S).$$

*Then*

$$\mathrm{score}_F(a) = 1,\ \mathrm{score}_F(b) = 3,\ \mathrm{score}_F(c) = 1,\ \mathrm{score}_F(d) = 1, \quad \Rightarrow \quad \star(F) = b \in S.$$

*Interpretation: from a* collection of committees *(an input in $\mathscr{P}^2(S)$), the organization deterministically picks a* single person *(output in $S = \mathscr{P}^0(S)$), so $(m,n) = (2,0)$.*

### 1.3. MetaGraph (Graph of Graphs)

Graph theory investigates mathematical structures consisting of vertices and edges to model relationships and connectivity [11, 13]. A *MetaGraph* is a graph whose vertices are themselves graphs, and whose edges represent specified relations between those graphs (cf. [58, 58–60]). MetaGraphs are known to have various applications in fields such as network theory and biology (cf. [61–64]).

**Definition 1.15** (Metagraph (graph of graphs))**.** *(cf.[64, 65]) Fix a nonempty universe $\mathfrak{G}$ of finite graphs (undirected, loopless by default) and a nonempty family of binary relations*

$$\mathscr{R} \subseteq \mathscr{P}\big(\mathfrak{G} \times \mathfrak{G}\big).$$

*A* metagraph over $(\mathfrak{G},\mathscr{R})$ *is a directed, labelled multigraph*

$$M = (V,E,s,t,\lambda)$$

*with*

$$V \subseteq \mathfrak{G}, \qquad s,t : E \to V, \qquad \lambda : E \to \mathscr{R},$$

*satisfying the incidence constraint*

$$\forall e \in E : \ \big(s(e),t(e)\big) \in \lambda(e).$$

*Elements of $V$ are* meta-vertices *(each is a graph $G \in \mathfrak{G}$). For $e \in E$ with $\lambda(e) = R$, we write $s(e) \xrightarrow{R} t(e)$ and call $e$ a* meta-edge. *If $\mathscr{R} = \{R\}$ is a singleton, labels may be omitted. If every $R \in \mathscr{R}$ is symmetric, $M$ can be viewed as an undirected labelled multigraph.*

**Example 1.16** (Real-life Metagraph: Transportation Networks)**.** *Consider a universe $\mathfrak{G}$ where each element $G \in \mathfrak{G}$ is a city-level transportation graph: vertices are bus or train stations, and edges are local routes. Define $\mathscr{R}$ as the family of relations "shares at least one transfer station." A metagraph $M = (V,E,s,t,\lambda)$ then has*

- *meta-vertices $V \subseteq \mathfrak{G}$, each representing a city's local transport network;*
- *a meta-edge $G_1 \xrightarrow{R} G_2$ whenever two cities $G_1, G_2$ are connected by at least one transfer station (e.g., a high–speed rail hub).*

*Thus, $M$ models a network of transportation networks: a* graph of graphs.

### 1.4. Iterated MetaGraph(Graph of Graph of ... of Graph)

An Iterated MetaGraph is a graph whose vertices are metagraphs, recursively extending graph-of-graphs structure to multiple hierarchical levels [66].

**Definition 1.17** (Unit metagraph embedding)**.** *[66] For $X \in \mathfrak{G}$ define the* unit metagraph

$$\mathsf{U}(X) := \big(\{X\}, \emptyset, \_, \_, \_\big).$$

*This gives an injective map $\mathsf{U} : \mathfrak{G} \hookrightarrow \mathrm{Obj}\big(\mathrm{Meta}(\mathfrak{G},\mathscr{R})\big)$.*

**Definition 1.18** (Relation lifting)**.** *[66] Given $\mathscr{R}$ on $\mathfrak{G}$, define its* lift $\mathscr{R}^{\uparrow}$ *on finite metagraphs over $(\mathfrak{G},\mathscr{R})$ by*

$$\forall R \in \mathscr{R}, \quad (M_1,M_2) \in R^{\uparrow} \iff \exists x \in V(M_1), y \in V(M_2) : \ (x,y) \in R.$$

*Set $\mathscr{R}^{\uparrow} := \{R^{\uparrow} : R \in \mathscr{R}\}$.*

**Definition 1.19** (Iterated object and relation universes). *Define recursively for $t \in \mathbb{N}_0$:*

$$\mathfrak{G}^{(0)} := \mathfrak{G}, \qquad \mathscr{R}^{(0)} := \mathscr{R},$$

$$\mathfrak{G}^{(t+1)} := \left\{ \textit{finite metagraphs over } \left(\mathfrak{G}^{(t)}, \mathscr{R}^{(t)}\right) \right\}, \qquad \mathscr{R}^{(t+1)} := \left(\mathscr{R}^{(t)}\right)^{\uparrow}.$$

**Definition 1.20** (Iterated MetaGraph of depth $t$). *For $t \in \mathbb{N}_0$, an* iterated metagraph of depth $t$ *is a metagraph*

$$M^{(t)} = (V^{(t)}, E^{(t)}, s^{(t)}, t^{(t)}, \lambda^{(t)})$$

*over $\left(\mathfrak{G}^{(t)}, \mathscr{R}^{(t)}\right)$, i.e., $V^{(t)} \subseteq \mathfrak{G}^{(t)}$, $\lambda^{(t)} : E^{(t)} \to \mathscr{R}^{(t)}$ and*

$$\forall e \in E^{(t)} : \left(s^{(t)}(e), t^{(t)}(e)\right) \in \lambda^{(t)}(e).$$

**Example 1.21** (Real-life Iterated Metagraph: International Trade Alliances). *Let $\mathfrak{G}$ be graphs where vertices are companies and edges are trade agreements within a single country. A metagraph $M^{(1)}$ then has vertices as country-level trade graphs and edges representing bilateral trade agreements between countries. Now consider an* iterated metagraph $M^{(2)}$*, whose vertices are these national metagraphs $M^{(1)}$, and whose edges represent regional trade alliances (e.g., EU, NAFTA). Thus:*

- *Level* 0*: graphs of companies trading within a country.*
- *Level* 1*: metagraph of countries (each a graph of companies).*
- *Level* 2*: iterated metagraph of regional alliances (each a metagraph of countries).*

*This illustrates a* hierarchy of trade networks*, modeled naturally by iterated metagraphs.*

# 2. Main Results of this Paper

This section presents the main results established in this paper.

## 2.1. MetaStructure (Structure of Structure)

We first fix a general single-sorted, finitary *signature*

$$\Sigma = \left(\mathsf{Func}, \mathsf{Rel}, \mathsf{ar}_{\mathsf{Func}}, \mathsf{ar}_{\mathsf{Rel}}\right),$$

where Func (resp. Rel) is a set of function (resp. relation) symbols, and ar records arities. A (single-sorted) $\Sigma$-*structure* is

$$\mathbf{C} = \left(H, (f^{\mathbf{C}})_{f \in \mathsf{Func}}, (R^{\mathbf{C}})_{R \in \mathsf{Rel}}\right),$$

with carrier $H \neq \emptyset$, interpretations $f^{\mathbf{C}} : H^m \to H$ for each $f \in \mathsf{Func}$ of arity $m$, and relations $R^{\mathbf{C}} \subseteq H^r$ for each $R \in \mathsf{Rel}$ of arity $r$. Let $\mathrm{Str}_\Sigma$ denote the class of all $\Sigma$-structures.

**Definition 2.1** (MetaStructure over a fixed signature). *Fix $\Sigma$ as above. A* MetaStructure *("structure of structures") over $\Sigma$ is a pair*

$$\mathbb{M} = \left(U, (\Phi_\ell)_{\ell \in \Lambda}\right),$$

*where:*

- *$U$ is a nonempty set with $U \subseteq \mathrm{Str}_\Sigma$ (its elements are* objects *at level 0);*
- *for each label $\ell \in \Lambda$ of* meta-arity $k_\ell \in \mathbb{N}$*, the* meta-operation

  $$\Phi_\ell : U^{k_\ell} \longrightarrow U$$

  *is specified by uniform* carrier- and symbol-constructors*:*

  $$\begin{aligned} &\Gamma_\ell : (\mathbf{C}_1, \ldots, \mathbf{C}_{k_\ell}) \mapsto H_\ell && \textit{(new carrier } H_\ell \textit{ built functorially)};\\ &\forall f \in \mathsf{Func} : \quad f^{\Phi_\ell(\mathbf{C}_1,\ldots,\mathbf{C}_{k_\ell})} = \Lambda_\ell^f\left(f^{\mathbf{C}_1}, \ldots, f^{\mathbf{C}_{k_\ell}}\right);\\ &\forall R \in \mathsf{Rel} : \quad R^{\Phi_\ell(\mathbf{C}_1,\ldots,\mathbf{C}_{k_\ell})} = \Xi_\ell^R\left(R^{\mathbf{C}_1}, \ldots, R^{\mathbf{C}_{k_\ell}}\right), \end{aligned}$$

  *where $\Lambda_\ell^f$ and $\Xi_\ell^R$ are* uniform *recipes turning the symbols' interpretations on inputs into the symbol's interpretation on the output, over the new carrier $H_\ell$.*

*Moreover, each $\Phi_\ell$ is* isomorphism-invariant *(a.k.a. natural): if $\alpha_i : \mathbf{C}_i \cong \mathbf{D}_i$ for $1 \le i \le k_\ell$, then there is an induced isomorphism*

$$\Phi_\ell(\alpha_1, \ldots, \alpha_{k_\ell}) : \Phi_\ell(\mathbf{C}_1, \ldots, \mathbf{C}_{k_\ell}) \xrightarrow{\cong} \Phi_\ell(\mathbf{D}_1, \ldots, \mathbf{D}_{k_\ell})$$

*commuting with all interpretations of symbols of $\Sigma$.*

**Remark 2.2** (Canonical meta-operations). *Three important $\Phi_\ell$ that will be used explicitly are:*

- Product $\Pi$*: for $(\mathbf{C}_1, \mathbf{C}_2)$ with carriers $H_1, H_2$,*

  $$\Gamma_\Pi(H_1, H_2) = H_1 \times H_2, \quad (f^{\mathbf{C}_1} \times f^{\mathbf{C}_2}) \textit{ componentwise on } (H_1 \times H_2)^m,$$

  *and for a relation $R$ of arity $r$,*

  $$R^{\mathbf{C}_1 \Pi \mathbf{C}_2} = \left\{ \left((x_1,y_1), \ldots, (x_r,y_r)\right) \mid (x_1, \ldots, x_r) \in R^{\mathbf{C}_1}, (y_1, \ldots, y_r) \in R^{\mathbf{C}_2} \right\}.$$

- Relational disjoint union $\uplus$ *(well-defined for purely relational* $\Sigma$*): for carriers* $H_1, H_2$,

  $$\Gamma_{\uplus}(H_1,H_2) = \{1\}\times H_1 \,\cup\, \{2\}\times H_2,$$

  *and for a relation R of arity r,*

  $$R^{\mathbf{C}_1 \uplus \mathbf{C}_2} \;=\; \big(\{1\}\times H_1\big)^r \cap \big(\{1\}\times R^{\mathbf{C}_1}\big) \;\cup\; \big(\{2\}\times H_2\big)^r \cap \big(\{2\}\times R^{\mathbf{C}_2}\big).$$

- Reduct/Expansion *w.r.t. sub-signatures (arity* 1*): forget or add symbols uniformly.*

*Each is evidently isomorphism-invariant.*

**Remark 2.3** (Level-0 vs. meta-level)**.** *A classical* $\Sigma$*-structure* $\mathbf{C}$ *(level-*0 *object) can be used as an element of* $U$*. Meta-operations combine such* structures *to create new* structures*, not elements of carriers.*

**Example 2.4** (Shared calendars (work schedules) as a MetaStructure)**.** *Fix a finite time grid H (e.g., 30-minute slots on a given day). Consider the single-sorted* calendar signature

$$\Sigma_{\text{cal}}: \quad \mathsf{Func} = \varnothing, \qquad \mathsf{Rel} = \{\text{Busy}\}, \quad \text{ar(Busy)} = 1.$$

*An object in* $U \subseteq \text{Str}_{\Sigma_{\text{cal}}}$ *is a structure*

$$\mathbf{C} = (H,\ \text{Busy}^{\mathbf{C}} \subseteq H),$$

*interpreted as a personal calendar: a slot* $h \in H$ *belongs to* $\text{Busy}^{\mathbf{C}}$ *iff the person is busy at h.*
***Meta-operation (overlay of calendars).*** *Define* $\Phi_{\cup} : U^2 \to U$ *by the uniform constructors*

$$\Gamma_{\cup}(H,H) = H, \qquad \Xi_{\cup}^{\text{Busy}}(B_1,B_2) := B_1 \cup B_2,$$

*so that*

$$\Phi_{\cup}(\mathbf{C}_1,\mathbf{C}_2) = \big(H,\ \text{Busy}^{\mathbf{C}_1} \cup \text{Busy}^{\mathbf{C}_2}\big).$$

*This is isomorphism-invariant (any bijection of the grid transports* $\cup$ *to* $\cup$*).*
***Concrete instance.*** *Let* $H = \{t_1,t_2,t_3,t_4\}$ *be four consecutive 30-minute slots (e.g., 09:00, 09:30, 10:00, 10:30). Suppose*

$$\text{Busy}^{\mathbf{C}_1} = \{t_1,t_3\}, \qquad \text{Busy}^{\mathbf{C}_2} = \{t_2,t_3\}.$$

*Then*

$$\Phi_{\cup}(\mathbf{C}_1,\mathbf{C}_2) = \big(H,\ \{t_1,t_2,t_3\}\big),$$

*which encodes the* combined busy time. *(Common free slots are* $H \setminus \{t_1,t_2,t_3\} = \{t_4\}$.*)*

**Example 2.5** (Transportation networks (road/rail overlays) as a MetaStructure)**.** *Let the single-sorted* graph signature *be*

$$\Sigma_{\text{gr}}: \quad \mathsf{Func} = \varnothing, \qquad \mathsf{Rel} = \{\text{Edge}\}, \quad \text{ar(Edge)} = 2.$$

*An object in* $U \subseteq \text{Str}_{\Sigma_{\text{gr}}}$ *is a directed network*

$$\mathbf{G} = (V,\ \text{Edge}^{\mathbf{G}} \subseteq V \times V).$$

*Think of V as stations/intersections and* $\text{Edge}^{\mathbf{G}}$ *as direct connections.*
***Meta-operation (relational disjoint union = overlay).*** *Define* $\Phi_{\uplus} : U^2 \to U$ *via the uniform constructors*

$$\Gamma_{\uplus}(V_1,V_2) := \{1\}\times V_1 \,\cup\, \{2\}\times V_2,$$

$$\Xi_{\uplus}^{\text{Edge}}(E_1,E_2) := \big(\{1\}\times E_1\big) \,\cup\, \big(\{2\}\times E_2\big),$$

*where* $\{i\}\times E = \{((i,u),(i,v)) \mid (u,v) \in E\}$*. Thus*

$$\Phi_{\uplus}(\mathbf{G}_1,\mathbf{G}_2) = \big(V_1 \uplus V_2,\ E_1 \uplus E_2\big),$$

*a canonical overlay that keeps each network as a labelled block. This construction is isomorphism-invariant (bijective relabellings of* $V_1, V_2$ *lift to* $\uplus$*).*
***Concrete instance.*** *Let*

$$\mathbf{G}_1:\ V_1 = \{a_1,a_2,a_3\}, \quad E_1 = \{(a_1,a_2),(a_2,a_3)\}$$

*(a simple line), and*

$$\mathbf{G}_2:\ V_2 = \{b_1,b_2\}, \quad E_2 = \{(b_1,b_2)\}$$

*(a short spur). Then*

$$\Phi_{\uplus}(\mathbf{G}_1,\mathbf{G}_2):\ V_1 \uplus V_2 = \{(1,a_1),(1,a_2),(1,a_3),(2,b_1),(2,b_2)\},$$

$$E_1 \uplus E_2 = \{((1,a_1),(1,a_2)),\ ((1,a_2),(1,a_3)),\ ((2,b_1),(2,b_2))\}.$$

*Interpretation: an* overlayed regional network *that preserves each city's routes (block labels* 1 *and* 2*). Additional connectors (inter-city links) may be added later as separate meta-operations that are likewise uniform (e.g., by a fixed rule that joins designated hubs).*

We first isolate a finitary signature for deterministic automata over a fixed finite alphabet $A$.

**Definition 2.6** (Automata signature $\Sigma_{\mathrm{dfa}}$)**.** *Let $A$ be a fixed finite alphabet. Define the single-sorted, finitary signature*

$$\Sigma_{\mathrm{dfa}} := \Big( \mathsf{Func},\ \mathsf{Rel} \Big),$$

*with*

$$\mathsf{Func} = \{\, \delta_a \mid a \in A \,\} \cup \{\, \mathsf{init} \,\}, \qquad \mathsf{Rel} = \{\, \mathsf{final} \,\}.$$

*The arities are* $\mathrm{ar}(\delta_a) = 1$, $\mathrm{ar}(\mathsf{init}) = 0$, *and* $\mathrm{ar}(\mathsf{final}) = 1$. *A $\Sigma_{\mathrm{dfa}}$–structure*

$$\mathbf{D} = \big( Q,\ (\delta_a^{\mathbf{D}})_{a\in A},\ q_0^{\mathbf{D}},\ \mathsf{final}^{\mathbf{D}} \ \big)$$

*is precisely a deterministic automaton over $A$: a nonempty state set $Q$, unary transition maps $\delta_a^{\mathbf{D}} : Q \to Q$, a distinguished initial state $q_0^{\mathbf{D}} \in Q$, and a unary relation $\mathsf{final}^{\mathbf{D}} \subseteq Q$ identifying final states.*

**Example 2.7** (MetaAutomata over $A$)**.** *Fix a nonempty universe $H$ of "atoms" and consider*

$$U_{\mathrm{dfa}} := \Big\{ \mathbf{D} \in \mathrm{Str}_{\Sigma_{\mathrm{dfa}}} \ \Big|\ Q := |\mathbf{D}| \subseteq H \Big\}.$$

*Define one binary meta–operation $\Phi_\otimes : U_{\mathrm{dfa}}^2 \to U_{\mathrm{dfa}}$, the* synchronous product. *For inputs*

$$\mathbf{D}_1 = (Q_1,\ (\delta_a^{(1)}),\ q_0^{(1)},\ \mathsf{final}^{(1)})$$

*and*

$$\mathbf{D}_2 = (Q_2,\ (\delta_a^{(2)}),\ q_0^{(2)},\ \mathsf{final}^{(2)})$$

*set*

$$\Phi_\otimes(\mathbf{D}_1,\mathbf{D}_2) \ := \ \Big(\ Q_1 \times Q_2,\ (\delta_a^{\otimes})_{a\in A},\ (q_0^{(1)},q_0^{(2)}),\ \mathsf{final}^{\otimes} \ \Big),$$

*where the constructors are* uniform*:*

$$\begin{aligned}
\Gamma_\otimes(Q_1,Q_2) &:= Q_1 \times Q_2,\\
\Lambda_\otimes^{\delta_a}\big(\delta_a^{(1)},\delta_a^{(2)}\big)(q_1,q_2) &:= \big(\delta_a^{(1)}(q_1),\ \delta_a^{(2)}(q_2)\big) \qquad (a \in A),\\
\Lambda_\otimes^{\mathsf{init}}\big(q_0^{(1)},q_0^{(2)}\big) &:= (q_0^{(1)},q_0^{(2)}),\\
\Xi_\otimes^{\mathsf{final}}\big(\mathsf{final}^{(1)},\mathsf{final}^{(2)}\big) &:= \mathsf{final}^{(1)} \times \mathsf{final}^{(2)}.
\end{aligned}$$

**Lemma 2.8** (MetaAutomata is a MetaStructure)**.** *With $U := U_{\mathrm{dfa}}$ and the single meta–operation $\Phi_\otimes$ from Example 2.7, the pair*

$$\mathbb{M}_{\mathrm{Auto}} = \big(U,\ (\Phi_\otimes)\big)$$

*is a MetaStructure in the sense of the Definition.*

*Proof.* **(Closure)** By construction $Q_1, Q_2 \subseteq H$ implies $Q_1 \times Q_2 \subseteq H \times H \subseteq H$ (after fixing a canonical injection $H \times H \hookrightarrow H$ once and for all), so the output carrier lies in $H$. Each symbol is interpreted by a well–typed, uniform recipe:

$$\delta_a^{\otimes} : (Q_1 \times Q_2) \to (Q_1 \times Q_2), \quad \mathsf{final}^{\otimes} \subseteq Q_1 \times Q_2, \quad (q_0^{(1)},q_0^{(2)}) \in Q_1 \times Q_2.$$

Hence $\Phi_\otimes(\mathbf{D}_1,\mathbf{D}_2) \in U$.

**(Isomorphism–invariance)** Let $\alpha_i : \mathbf{D}_i \cong \mathbf{D}_i'$ be automaton isomorphisms, i.e. bijections $\alpha_i : Q_i \to Q_i'$ such that

$$\alpha_i\big(\delta_a^{(i)}(q)\big) = \delta_a^{(i)\prime}\big(\alpha_i(q)\big), \qquad \alpha_i\big(q_0^{(i)}\big) = q_0^{(i)\prime}, \qquad q \in \mathsf{final}^{(i)} \ \Leftrightarrow\ \alpha_i(q) \in \mathsf{final}^{(i)\prime}.$$

Then

$$\alpha_1 \times \alpha_2 :\ Q_1 \times Q_2 \longrightarrow Q_1' \times Q_2'$$

is an isomorphism witnessing

$$\Phi_\otimes(\alpha_1,\alpha_2) :\ \Phi_\otimes(\mathbf{D}_1,\mathbf{D}_2) \ \xrightarrow{\ \cong\ }\ \Phi_\otimes(\mathbf{D}_1',\mathbf{D}_2'),$$

since for every $a \in A$ and $(q_1,q_2) \in Q_1 \times Q_2$,

$$\begin{aligned}
(\alpha_1 \times \alpha_2)\big(\delta_a^{\otimes}(q_1,q_2)\big) &= (\alpha_1 \times \alpha_2)\big(\delta_a^{(1)}(q_1),\delta_a^{(2)}(q_2)\big)\\
&= \big(\alpha_1(\delta_a^{(1)}(q_1)),\ \alpha_2(\delta_a^{(2)}(q_2))\big)\\
&= \big(\delta_a^{(1)\prime}(\alpha_1(q_1)),\ \delta_a^{(2)\prime}(\alpha_2(q_2))\big)\\
&= \delta_a^{\otimes\prime}\big((\alpha_1 \times \alpha_2)(q_1,q_2)\big),
\end{aligned}$$

and similarly for $\mathsf{init}$ and $\mathsf{final}$. Therefore Definition 2.1 holds. □

**Remark 2.9** (Language–theoretic semantics)**.** *If $L(\mathbf{D}) \subseteq A^*$ denotes the language recognized by* $\mathbf{D}$*, then*

$$L\big(\Phi_\otimes(\mathbf{D}_1,\mathbf{D}_2)\big) = L(\mathbf{D}_1) \ \cap \ L(\mathbf{D}_2).$$

*This identity is a concrete, verifiable invariant of the meta–operation $\Phi_\otimes$.*

**Definition 2.10** (Topological signature $\Sigma_{\mathrm{cl}}$)**.** *Fix a nonempty set $\Omega$ and a distinguished subset $X_0 \subseteq \Omega$. Let the carrier of every structure be $H := \mathscr{P}(\Omega)$. Define the signature*

$$\Sigma_{\mathrm{cl}} \ := \ \Big( \, \mathsf{Func} = \{\mathrm{cl}\}, \ \mathsf{Rel} = \{\mathsf{X}\} \, \Big),$$

*with* $\mathrm{ar}(\mathrm{cl}) = 1$*,* $\mathrm{ar}(\mathsf{X}) = 1$*. A $\Sigma_{\mathrm{cl}}$–structure is a pair*

$$\mathbf{T} = \big( \, H, \ \mathrm{cl}^{\mathbf{T}} : H \to H, \ \mathsf{X}^{\mathbf{T}} \subseteq H \, \big),$$

*interpreted as:* $\mathsf{X}^{\mathbf{T}} = \{X_0\}$ *is the singleton identifying the underlying set, and* $\mathrm{cl}^{\mathbf{T}}$ *is a Kuratowski closure on $X_0$, i.e. for every $A \subseteq X_0$,*

$$\mathrm{cl}(\varnothing) = \varnothing, \qquad A \subseteq \mathrm{cl}(A), \qquad \mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B), \qquad \mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A). \tag{1}$$

*(For $A \nsubseteq X_0$ we set $\mathrm{cl}(A) := \mathrm{cl}(A \cap X_0)$ so that the map is total on $H$.)*

**Example 2.11** (MetaTopology over the fixed base $X_0$)**.** *Let*

$$U_{\mathrm{top}} := \Big\{ \ \mathbf{T} \in \mathrm{Str}_{\Sigma_{\mathrm{cl}}} \ \Big| \ \mathsf{X}^{\mathbf{T}} = \{X_0\} \ \textit{ and } \ \mathrm{cl}^{\mathbf{T}} \textit{ satisfies } (1) \textit{ on } X_0 \ \Big\}.$$

*Define a binary meta–operation $\Phi_\vee : U_{\mathrm{top}}^2 \to U_{\mathrm{top}}$ (the* topological join/refinement*) by the uniform constructors*

$$\begin{aligned}
\Gamma_\vee(H,H) &:= H \quad (\textit{the carrier stays } H = \mathscr{P}(\Omega)),\\
\Xi_\vee^{\mathsf{X}}(\{X_0\},\{X_0\}) &:= \{X_0\},\\
\Lambda_\vee^{\mathrm{cl}}(\mathrm{cl}_1,\mathrm{cl}_2)(A) &:= \mathrm{cl}_1(A) \ \cap \ \mathrm{cl}_2(A) \qquad (A \subseteq \Omega).
\end{aligned}$$

*We write $\Phi_\vee(\mathbf{T}_1,\mathbf{T}_2) =: \mathbf{T}_1 \vee \mathbf{T}_2$.*

**Lemma 2.12** (MetaTopology is a MetaStructure)**.** *With $U := U_{\mathrm{top}}$ and $\Phi_\vee$ from Example 2.11, the pair*

$$\mathbb{M}_{\mathrm{Topo}} = \big(U, \ (\Phi_\vee)\big)$$

*is a MetaStructure in the sense of Definition 2.1. Moreover, if $\tau(\mathbf{T})$ denotes the topology on $X_0$ corresponding to* $\mathrm{cl}^{\mathbf{T}}$*, then*

$$\tau(\mathbf{T}_1 \vee \mathbf{T}_2) \ = \ \tau(\mathbf{T}_1) \ \vee \ \tau(\mathbf{T}_2)$$

*(the lattice join of topologies on $X_0$).*

*Proof.* **(Closure)** Let $\mathbf{T}_i = (H, \mathrm{cl}_i, \{X_0\}) \in U$ for $i = 1,2$. Define $\mathrm{cl}_\vee(A) := \mathrm{cl}_1(A) \cap \mathrm{cl}_2(A)$ for all $A \subseteq \Omega$. For $A \subseteq X_0$, each Kuratowski axiom holds:

$$\begin{aligned}
\mathrm{cl}_\vee(\varnothing) &= \mathrm{cl}_1(\varnothing) \cap \mathrm{cl}_2(\varnothing) = \varnothing \cap \varnothing = \varnothing,\\
&A \subseteq \mathrm{cl}_1(A) \text{ and } A \subseteq \mathrm{cl}_2(A) \implies A \subseteq \mathrm{cl}_1(A) \cap \mathrm{cl}_2(A) = \mathrm{cl}_\vee(A),\\
\mathrm{cl}_\vee(A \cup B) &= \mathrm{cl}_1(A \cup B) \cap \mathrm{cl}_2(A \cup B)\\
&= (\mathrm{cl}_1(A) \cup \mathrm{cl}_1(B)) \ \cap \ (\mathrm{cl}_2(A) \cup \mathrm{cl}_2(B))\\
&= \big(\mathrm{cl}_1(A) \cap \mathrm{cl}_2(A)\big) \ \cup \ \big(\mathrm{cl}_1(A) \cap \mathrm{cl}_2(B)\big) \ \cup \ \big(\mathrm{cl}_1(B) \cap \mathrm{cl}_2(A)\big) \ \cup \ \big(\mathrm{cl}_1(B) \cap \mathrm{cl}_2(B)\big)\\
&\supseteq \big(\mathrm{cl}_1(A) \cap \mathrm{cl}_2(A)\big) \ \cup \ \big(\mathrm{cl}_1(B) \cap \mathrm{cl}_2(B)\big) = \mathrm{cl}_\vee(A) \cup \mathrm{cl}_\vee(B).
\end{aligned}$$

For the reverse inclusion, recall that closed sets in $\tau(\mathbf{T}_1) \vee \tau(\mathbf{T}_2)$ are exactly intersections $C_1 \cap C_2$ with $C_i$ closed in $\tau(\mathbf{T}_i)$; hence for every $A \subseteq X_0$,

$$\mathrm{cl}_\vee(A) = \bigcap \{ C_1 \cap C_2 \mid A \subseteq C_1 \cap C_2, \ C_i \text{ closed in } \tau(\mathbf{T}_i) \} = \mathrm{cl}_1(A) \cap \mathrm{cl}_2(A).$$

Applying this equality to $A$ and to $B$ gives

$$\mathrm{cl}_\vee(A) \cup \mathrm{cl}_\vee(B) = \bigcap_{A \subseteq C_1 \cap C_2} (C_1 \cap C_2) \ \cup \ \bigcap_{B \subseteq D_1 \cap D_2} (D_1 \cap D_2) = \bigcap_{A \subseteq C_1 \cap C_2, \ B \subseteq D_1 \cap D_2} \big((C_1 \cap C_2) \cup (D_1 \cap D_2)\big),$$

and since $A \cup B \subseteq (C_1 \cup D_1) \cap (C_2 \cup D_2)$ with $(C_i \cup D_i)$ closed, the right–hand side contains $\mathrm{cl}_\vee(A \cup B)$, yielding the equality $\mathrm{cl}_\vee(A \cup B) = \mathrm{cl}_\vee(A) \cup \mathrm{cl}_\vee(B)$. Finally,

$$\mathrm{cl}_\vee\big(\mathrm{cl}_\vee(A)\big) = \mathrm{cl}_1\big(\mathrm{cl}_1(A) \cap \mathrm{cl}_2(A)\big) \ \cap \ \mathrm{cl}_2\big(\mathrm{cl}_1(A) \cap \mathrm{cl}_2(A)\big) \subseteq \mathrm{cl}_1(\mathrm{cl}_1(A)) \cap \mathrm{cl}_2(\mathrm{cl}_2(A)) = \mathrm{cl}_\vee(A),$$

and the reverse inclusion follows from extensivity, so idempotence holds. Thus $\mathbf{T}_1 \vee \mathbf{T}_2 \in U$.

**(Isomorphism–invariance)** Any bijection $\alpha : \Omega \to \Omega$ induces $\alpha^\sharp : \mathscr{P}(\Omega) \to \mathscr{P}(\Omega)$, $A \mapsto \alpha[A]$. Transporting $\mathrm{cl}_i$ to $\mathrm{cl}'_i$ by $\mathrm{cl}'_i := \alpha^\sharp \circ \mathrm{cl}_i \circ (\alpha^\sharp)^{-1}$ and $X_0$ to $\alpha[X_0]$ commutes with the constructor:

$$\alpha^\sharp \circ \Lambda_\vee^{\mathrm{cl}}(\mathrm{cl}_1,\mathrm{cl}_2) \circ (\alpha^\sharp)^{-1} = \Lambda_\vee^{\mathrm{cl}}(\mathrm{cl}'_1,\mathrm{cl}'_2).$$

Since $\mathsf{X}$ is transported as $\{X_0\} \mapsto \{\alpha[X_0]\}$, the induced map is an isomorphism between outputs, as required by Definition 2.1. The lattice statement follows from the closed–set characterization used above. □

### 2.2. iterated MetaStructure(Structure of Structure of ... of Structure)

An Iterated MetaStructure recursively applies MetaStructure construction, forming successive layers where structures of structures create deeper hierarchical meta-levels.

**Definition 2.13** (Iterated MetaStructure of depth $t$)**.** *An* Iterated MetaStructure of depth $t$ *over* $\Sigma$ *is any MetaStructure* $\mathfrak{M}^{(t)}$ *of height* $t$. *When* $s<t$, *we* lift *a height-s MetaStructure* $\mathfrak{M}^{(s)} = (U^{(s)}, \{\odot_i\}, \{\mathscr{S}_j\})$ *to height t by*

$$\iota_{s\to t}:\ U^{(s)} \xrightarrow{\mathsf{U}_\Sigma^{t-s}} U^{(t)} := \mathsf{U}_\Sigma^{t-s}\big(U^{(s)}\big),$$

*and, for each* $\odot_i : (\mathsf{E}_\Sigma^{m_i})^{k_i} \to \mathscr{P}^{n_i}(\mathsf{E}_\Sigma^{n_i})$*, defining its lift*

$$\odot_i^{\uparrow}:\ \big(\mathsf{E}_\Sigma^{m_i+t-s}\big)^{k_i} \longrightarrow \mathscr{P}^{n_i}\big(\mathsf{E}_\Sigma^{n_i+t-s}\big),\quad \odot_i^{\uparrow}\big(\mathsf{U}_\Sigma^{t-s}(x_1),\ldots,\mathsf{U}_\Sigma^{t-s}(x_{k_i})\big) := \mathsf{U}_\Sigma^{t-s}\big(\odot_i(x_1,\ldots,x_{k_i})\big),$$

*and similarly for relations* $\mathscr{S}_j^{\uparrow} := \big(\mathsf{U}_\Sigma^{t-s}\big)^{\times \ell_j}(\mathscr{S}_j)$.

**Example 2.14** (Iterated MetaAlgebra: products of algebras at depth $t$)**.** *Fix the single–sorted algebraic signature* $\Sigma_{\text{alg}} = \{\cdot\}$ *with* $\operatorname{ar}(\cdot)=2$ *(semigroups). Let $H$ be a universe of atoms and define the ladder* $\mathsf{E}^0_{\Sigma_{\text{alg}}}(H) := H$ *and*

$$\mathsf{E}^{t+1}_{\Sigma_{\text{alg}}}(H) := \big\{\mathbf{A} = (A, \cdot^{\mathbf{A}}) \ \big|\ A \subseteq \mathsf{E}^{t}_{\Sigma_{\text{alg}}}(H),\ \cdot^{\mathbf{A}} : A^2 \to A\big\}.$$

*Set* $U^{(t)}_{\text{alg}} := \mathsf{E}^{t}_{\Sigma_{\text{alg}}}(H)$*. For* $\mathbf{A},\mathbf{B} \in U^{(t)}_{\text{alg}}$*, define the meta–operation (componentwise product)*

$$\Phi^{(t)}_{\Pi}(\mathbf{A},\mathbf{B}) := \big(A\times B,\ \cdot^{\mathbf{A}\times\mathbf{B}}\big),\qquad (a_1,b_1)\cdot^{\mathbf{A}\times\mathbf{B}}(a_2,b_2) := \big(a_1\cdot^{\mathbf{A}} a_2,\ b_1\cdot^{\mathbf{B}} b_2\big).$$

Closure: $A,B \subseteq \mathsf{E}^{t-1}$ *implies* $A\times B \subseteq \mathsf{E}^{t-1}\times\mathsf{E}^{t-1}$*; fix a pairing injection* $\langle\ ,\ \rangle$ *to identify* $A\times B \subseteq \mathsf{E}^{t-1}$*, hence* $\Phi^{(t)}_{\Pi}(\mathbf{A},\mathbf{B}) \in U^{(t)}_{\text{alg}}$. Associativity: *for all* $a_i \in A$*,* $b_i \in B$*,*

$$\big((a_1,b_1)\cdot(a_2,b_2)\big)\cdot(a_3,b_3) = (a_1\cdot(a_2\cdot a_3),\ b_1\cdot(b_2\cdot b_3)) = (a_1,b_1)\cdot\big((a_2,b_2)\cdot(a_3,b_3)\big).$$

Lift compatibility (Definition 2.13): *for* $1\le s<t$*,*

$$\Phi^{(t)}_{\Pi}\big(\mathsf{U}^{t-s}_{\Sigma_{\text{alg}}}(\mathbf{A}),\mathsf{U}^{t-s}_{\Sigma_{\text{alg}}}(\mathbf{B})\big) = \mathsf{U}^{t-s}_{\Sigma_{\text{alg}}}\big(\Phi^{(s)}_{\Pi}(\mathbf{A},\mathbf{B})\big),$$

*since* $\mathsf{U}_{\Sigma_{\text{alg}}}$ *wraps elements in singletons and preserves componentwise formulas.*

Concrete numbers ($t=1$): *for groups* $C_2=\{\bar 0,\bar 1\}$*,* $C_3=\{\bar 0,\bar 1,\bar 2\}$ *(additive),*

$$(\bar 1,\bar 2)+(\bar 1,\bar 1) = (\bar 0,\bar 0)\ \ \text{in}\ \ C_2\times C_3 \cong C_6.$$

**Example 2.15** (Iterated MetaProbability: product and mixture at depth $t$)**.** *Use a two–sorted finite–probability signature* $\Sigma_{\text{fp}}$ *with outcomes* $\mathsf{Out}$ *and values* $\mathsf{Val}$*; structures are* $\mathbf{P}=(\Omega,p)$ *with $\Omega$ finite nonempty and* $p:\Omega\to[0,1]$*,* $\sum_{\omega\in\Omega} p(\omega)=1$*. Define* $\mathsf{E}^0_{\Sigma_{\text{fp}}}(H) := H$ *and*

$$\mathsf{E}^{t+1}_{\Sigma_{\text{fp}}}(H) := \big\{(\Omega,p)\ \big|\ \Omega\subseteq \mathsf{E}^{t}_{\Sigma_{\text{fp}}}(H),\ p \text{ pmf on } \Omega\big\},\quad U^{(t)}_{\text{fp}} := \mathsf{E}^{t}_{\Sigma_{\text{fp}}}(H).$$

Independent product (meta–operation):

$$\Phi^{(t)}_{\boxtimes}\big((\Omega_1,p_1),(\Omega_2,p_2)\big) := \big(\Omega_1\times\Omega_2,\ p_1\otimes p_2\big),\quad (p_1\otimes p_2)(\omega_1,\omega_2) := p_1(\omega_1)p_2(\omega_2).$$

*Then*

$$\sum_{\Omega_1\times\Omega_2}(p_1\otimes p_2) = \Big(\sum_{\Omega_1} p_1\Big)\Big(\sum_{\Omega_2} p_2\Big) = 1,$$

*so* $\Phi^{(t)}_{\boxtimes}(\mathbf{P}_1,\mathbf{P}_2) \in U^{(t)}_{\text{fp}}$. Mixture (meta–operation, $\lambda\in[0,1]$):

$$\Phi^{(t)}_{\text{mix},\lambda}\big((\Omega_1,p_1),(\Omega_2,p_2)\big) := \big(\{1\}\times\Omega_1\cup\{2\}\times\Omega_2,\ p_{\text{mix}}\big),$$
$$p_{\text{mix}}(1,\omega_1) := \lambda\, p_1(\omega_1),\quad p_{\text{mix}}(2,\omega_2) := (1-\lambda)\, p_2(\omega_2),$$

*and*

$$\sum p_{\text{mix}} = \lambda\sum_{\Omega_1} p_1 + (1-\lambda)\sum_{\Omega_2} p_2 = \lambda + (1-\lambda) = 1.$$

Lift compatibility: *for* $1\le s<t$*,*

$$\Phi^{(t)}_{\boxtimes}\big(\mathsf{U}^{t-s}_{\Sigma_{\text{fp}}}(\mathbf{P}_1),\mathsf{U}^{t-s}_{\Sigma_{\text{fp}}}(\mathbf{P}_2)\big) = \mathsf{U}^{t-s}_{\Sigma_{\text{fp}}}\big(\Phi^{(s)}_{\boxtimes}(\mathbf{P}_1,\mathbf{P}_2)\big),$$
$$\Phi^{(t)}_{\text{mix},\lambda}\big(\mathsf{U}^{t-s}_{\Sigma_{\text{fp}}}(\mathbf{P}_1),\mathsf{U}^{t-s}_{\Sigma_{\text{fp}}}(\mathbf{P}_2)\big) = \mathsf{U}^{t-s}_{\Sigma_{\text{fp}}}\big(\Phi^{(s)}_{\text{mix},\lambda}(\mathbf{P}_1,\mathbf{P}_2)\big),$$

*since* $\mathsf{U}_{\Sigma_{\text{fp}}}$ *wraps outcomes as singletons and preserves the pointwise formulas.*

Concrete numbers ($t=1$): $\Omega_1=\{a,b\}$ *with* $p_1(a)=\frac13$*,* $p_1(b)=\frac23$*;* $\Omega_2=\{x,y\}$ *with* $p_2(x)=\frac12$*,* $p_2(y)=\frac12$*:*

$$(p_1\otimes p_2)(a,y) = \tfrac13\cdot\tfrac12 = \tfrac16,\qquad \sum_{\Omega_1\times\Omega_2}(p_1\otimes p_2) = 1.$$

*For* $\lambda=\frac14$*,*

$$p_{\text{mix}}(1,a) = \tfrac14\cdot\tfrac13 = \tfrac1{12},\quad p_{\text{mix}}(2,y) = \tfrac34\cdot\tfrac12 = \tfrac38,\quad \sum p_{\text{mix}} = 1.$$

**Theorem 2.16** (Iteration strictly generalizes)**.** *For each $t > s \geq 0$, the lift $\iota_{s\to t}$ embeds the category of height-s MetaStructures (with homomorphisms preserving meta–operations/relations) fully and faithfully into the category of height-t MetaStructures. In particular, Iterated MetaStructure strictly generalizes MetaStructure.*

*Proof. Full/faithful:* Given homomorphisms at height $s$, lifting them componentwise by $\mathsf{U}_{\Sigma}^{t-s}$ yields homomorphisms at height $t$; conversely, any height-$t$ morphism between lifted carriers restricts (via the singleton carriers of $\mathsf{U}_{\Sigma}^{t-s}$) to a unique height-$s$ morphism. *Strictness:* There exist height-$t$ MetaStructures whose carriers contain nontrivial $\Sigma$–structures on level $t-1$ (e.g. two distinct nonisomorphic level-$(t-1)$ objects in $U^{(t)}$), which cannot live at any smaller height $s < t$ by definition of $\mathsf{E}_{\Sigma}^{t}$. □

Recall the iterated metagraph construction from the Preliminaries: $\mathfrak{G}^{(0)} := \mathfrak{G}$, $\mathscr{R}^{(0)} := \mathscr{R}$ and, inductively, $\mathfrak{G}^{(t+1)} := \{$finite metagraphs over $(\mathfrak{G}^{(t)}, \mathscr{R}^{(t)})\}$ $\mathscr{R}^{(t+1)} := (\mathscr{R}^{(t)})^{\uparrow}$.

**Theorem 2.17** (Iterated MetaGraph $\Rightarrow$ Iterated MetaStructure)**.** *Let $M^{(t)}$ be an iterated metagraph of depth t over $(\mathfrak{G}^{(t)}, \mathscr{R}^{(t)})$. With $\Sigma_{\mathrm{gr}}$ and H, the assignment*

$$U^{(t)} := V\big(M^{(t)}\big) \subseteq \mathsf{E}_{\Sigma_{\mathrm{gr}}}^{t}(H), \qquad \mathscr{S}_R := \{(X,Y) \in U^{(t)} \times U^{(t)} \mid X \xrightarrow{R} Y \text{ in } M^{(t)}\},$$

*turns $M^{(t)}$ into an Iterated MetaStructure $\mathfrak{M}_{M^{(t)}}^{(t)}$ of depth t.*

*Proof.* By induction on $t$. The base $t{=}1$ is the Theorem. Suppose the claim holds for $t$. A vertex of $M^{(t+1)}$ is a metagraph over $(\mathfrak{G}^{(t)}, \mathscr{R}^{(t)})$; by the inductive hypothesis it is a height-$t$ MetaStructure, i.e. an element of $\mathsf{E}_{\Sigma_{\mathrm{gr}}}^{t}(H)$. Hence $U^{(t+1)} \subseteq \mathsf{E}_{\Sigma_{\mathrm{gr}}}^{t+1}(H)$ and relations $\mathscr{S}_R$ are defined exactly as in the $t{=}1$ case but now on level $t{+}1$. The axioms (MS1)–(MS3) are verified verbatim. Thus $\mathfrak{M}_{M^{(t+1)}}^{(t+1)}$ is a height-$(t{+}1)$ MetaStructure. □

## 2.3. Meta-HyperStructure(HyperStructure of HyperStructure)

Fix once and for all a (single–sorted, finitary) first–order signature

$$\Sigma = \big(\mathsf{Func}, \mathsf{Rel}, \mathsf{ar}_{\mathsf{Func}}, \mathsf{ar}_{\mathsf{Rel}}\big),$$

where each $f \in \mathsf{Func}$ has arity $\mathrm{ar}(f) \in \mathbb{N}_0$ and each $R \in \mathsf{Rel}$ has arity $\mathrm{ar}(R) \in \mathbb{N}$. A $\Sigma$–*structure* $\mathbf{C}$ is

$$\mathbf{C} = \big(|\mathbf{C}|,\ (f^{\mathbf{C}})_{f\in\mathsf{Func}},\ (R^{\mathbf{C}})_{R\in\mathsf{Rel}}\big)$$

with $|\mathbf{C}| \neq \varnothing$, $f^{\mathbf{C}} : |\mathbf{C}|^{\mathrm{ar}(f)} \to |\mathbf{C}|$ and $R^{\mathbf{C}} \subseteq |\mathbf{C}|^{\mathrm{ar}(R)}$.

For a set $X$, write $\mathscr{P}(X)$ for its powerset and inductively $\mathscr{P}_1(X) := \mathscr{P}(X)$ and $\mathscr{P}_{n+1}(X) := \mathscr{P}\big(\mathscr{P}_n(X)\big)$. We use $\mathscr{P}^0(X) := X$ and $\mathscr{P}^1(X) := \mathscr{P}(X)$ when unifying types.

A *hyperoperation on X of arity m* is a map $\circ : X^m \to \mathscr{P}(X)$. A *family of hyperoperations on* $X$ indexed by a finite set $Q$ is $\circ = (\circ^{(q)})_{q\in Q}$ with $\circ^{(q)} : X^{m_q} \to \mathscr{P}(X)$.

**Definition 2.18** ($\Sigma$–hyperstructure)**.** *A* $\Sigma$–hyperstructure *is a tuple*

$$\mathbf{H} = \Big(|\mathbf{H}|,\ (f^{\mathbf{H}})_{f\in\mathsf{Func}},\ (R^{\mathbf{H}})_{R\in\mathsf{Rel}},\ (\circ_{\mathbf{H}}^{(q)})_{q\in Q}\Big)$$

*where:*

- $|\mathbf{H}| \neq \varnothing$;
- *for each* $f \in \mathsf{Func}$, $f^{\mathbf{H}} : |\mathbf{H}|^{\mathrm{ar}(f)} \to |\mathbf{H}|$;
- *for each* $R \in \mathsf{Rel}$, $R^{\mathbf{H}} \subseteq |\mathbf{H}|^{\mathrm{ar}(R)}$;
- $(\circ_{\mathbf{H}}^{(q)})_{q\in Q}$ *is a specified finite family of hyperoperations on* $|\mathbf{H}|$.

*A morphism $\phi : \mathbf{H} \to \mathbf{H}'$ is a bijection $\phi : |\mathbf{H}| \to |\mathbf{H}'|$ preserving the f's, the R's, and each hyperoperation:*

$$\phi\big(f^{\mathbf{H}}(\vec{x})\big) = f^{\mathbf{H}'}\big(\phi(\vec{x})\big), \quad \vec{x} \in R^{\mathbf{H}} \Leftrightarrow \phi(\vec{x}) \in R^{\mathbf{H}'}, \quad \phi\big(\circ_{\mathbf{H}}^{(q)}(\vec{A})\big) = \circ_{\mathbf{H}'}^{(q)}\big(\phi(\vec{A})\big).$$

*Denote by* $\mathrm{HypStr}_{\Sigma}$ *the class of* $\Sigma$*–hyperstructures (up to isomorphism).*

Intuitively, objects are *hyperstructures*, and meta–*hyper*operations map tuples of hyperstructures to *sets* of hyperstructures, with all internal ingredients built uniformly.

**Definition 2.19** (Meta–HyperStructure over $\Sigma$)**.** *A* Meta–HyperStructure *over* $\Sigma$ *is a pair*

$$\mathbb{MH} = \Big(U,\ (\Phi_\ell)_{\ell\in\Lambda}\Big),$$

*where:*

- $U \subseteq \mathrm{HypStr}_{\Sigma}$ *is nonempty (objects);*

- *for each label* $\ell \in \Lambda$ *with meta–arity* $k_\ell \in \mathbb{N}$ *there is a* meta–hyperoperation

$$\Phi_\ell : U^{k_\ell} \longrightarrow \mathscr{P}(U)$$

*specified by the following* uniform constructors. *Given inputs* $\mathbf{H}_1,\dots,\mathbf{H}_{k_\ell} \in U$ *with carriers* $H_i := |\mathbf{H}_i|$, *operations* $(f^{\mathbf{H}_i})$, *relations* $(R^{\mathbf{H}_i})$ *and families* $(\circ^{(q)}_{\mathbf{H}_i})_{q\in Q}$, *every output* $\mathbf{K} \in \Phi_\ell(\mathbf{H}_1,\dots,\mathbf{H}_{k_\ell})$ *is built by:*

$$\begin{array}{ll}
\textit{(carrier)} & |\mathbf{K}| = \Gamma_\ell(H_1,\dots,H_{k_\ell}),\\
\textit{(operations)} & f^{\mathbf{K}} = \Lambda^f_\ell\big(f^{\mathbf{H}_1},\dots,f^{\mathbf{H}_{k_\ell}}\big) \quad (f \in \mathsf{Func}),\\
\textit{(relations)} & R^{\mathbf{K}} = \Xi^R_\ell\big(R^{\mathbf{H}_1},\dots,R^{\mathbf{H}_{k_\ell}}\big) \quad (R \in \mathsf{Rel}),\\
\textit{(hyperops)} & \circ^{(q)}_{\mathbf{K}} = \Pi^q_\ell\big(\circ^{(q)}_{\mathbf{H}_1},\dots,\circ^{(q)}_{\mathbf{H}_{k_\ell}}\big) \quad (q \in Q),
\end{array}$$

*where* $\Gamma_\ell, \Lambda^f_\ell, \Xi^R_\ell, \Pi^q_\ell$ *are fixed* functorial recipes *(independent of the particular inputs) ensuring* $\mathbf{K} \in \mathrm{HypStr}_\Sigma$.

*Axioms:*

$$\begin{array}{ll}
\textbf{\textit{(MH1) Closure:}} & \Phi_\ell(\mathbf{H}_1,\dots,\mathbf{H}_{k_\ell}) \subseteq U.\\
\textbf{\textit{(MH2) Natural/iso–invariant:}} & \text{if } \alpha_i : \mathbf{H}_i \cong \mathbf{H}'_i \ (1 \le i \le k_\ell),\\
& \text{then there is a bijection } \Phi_\ell(\alpha_1,\dots,\alpha_{k_\ell}) : \Phi_\ell(\vec{\mathbf{H}}) \xrightarrow{\ \cong\ } \Phi_\ell(\vec{\mathbf{H}}')\\
& \text{that is an isomorphism of outputs, componentwise for all symbols and hyperops.}
\end{array}$$

**Remark 2.20** (Typical uniform constructors)**.** *Common choices include:*

- *product on carriers,* $(\Gamma_\Pi)(H_1,H_2) = H_1 \times H_2$, *with componentwise lifting of* $f$ *and* tensor *lifting of relations; for hyperops,*

$$\Pi^q_\Pi\Big(\circ^{(q)}_1, \circ^{(q)}_2\Big)(\vec{A})$$

$$= \big\{\, (x_1,x_2) \;\big|\; x_i \in \circ^{(q)}_i\big(\mathrm{pr}_i(\vec{A})\big)\ (i=1,2) \,\big\};$$

- *disjoint union on carriers with block–diagonal relations and hyperops acting in each block;*
- *reduct/expansion with respect to sub–/super–signatures (meta–arity* 1*).*

*All are isomorphism–invariant by construction.*

**Example 2.21** (Block–sum of hypermagmas (disjoint union))**.** *Let* $\Sigma_{\mathrm{hm}}$ *be the single–sorted signature with no ordinary function/relation symbols and one binary* hyperoperation symbol $\star$. *A* $\Sigma_{\mathrm{hm}}$*–hyperstructure is a* hypermagma $\mathbf{H} = (H, \star_{\mathbf{H}})$ *with* $\star_{\mathbf{H}} : H \times H \to \mathscr{P}(H)$. *Let* $U$ *be any nonempty class of such hypermagmas (objects).*
*Define a binary meta–hyperoperation* $\Phi_\uplus : U^2 \to \mathscr{P}(U)$ *that returns the* single *output hypermagma on the disjoint union carrier:*

$$\Phi_\uplus\big((H_1,\star_1),(H_2,\star_2)\big) \ := \ \Big\{ \ \big(H_1 \uplus H_2,\ \star_\uplus\big) \ \Big\},$$

*where, writing elements as* $(i,x)$ *with* $i \in \{1,2\}$ *and* $x \in H_i$*,*

$$\star_\uplus\big((i,x),(j,y)\big) \ := \ \begin{cases} \{1\} \times \star_1(x,y), & (i,j) = (1,1),\\ \{2\} \times \star_2(x,y), & (i,j) = (2,2),\\ \varnothing, & i \neq j. \end{cases}$$

Uniform constructors (Definition 2.19):

$$\Gamma_\uplus(H_1,H_2) = H_1 \uplus H_2, \qquad \Pi^\star_\uplus(\star_1,\star_2) = \star_\uplus \qquad (\Lambda, \Xi \text{ are vacuous}).$$

(MH1) Closure: $\star_\uplus$ *maps* $(H_1 \uplus H_2)^2$ *into* $\mathscr{P}(H_1 \uplus H_2)$, *so the output lies in* $U$. (MH2) Naturality: *if* $\alpha_i : (H_i,\star_i) \cong (H'_i,\star'_i)$, *then* $\alpha_1 \uplus \alpha_2$ *is an isomorphism*

$$\big(H_1 \uplus H_2, \star_\uplus\big) \ \xrightarrow{\ \cong\ } \ \big(H'_1 \uplus H'_2, \star'_\uplus\big),$$

*since each block is transported componentwise and cross–block outputs remain* $\varnothing$. *Thus* $\big(U, \Phi_\uplus\big)$ *is a Meta–HyperStructure.*

**Example 2.22** (Product & coproduct meta–combination (set–valued output))**.** *Work again over* $\Sigma_{\mathrm{hm}}$ *and the same object class* $U$. *Define a binary meta–hyperoperation* $\Phi_{\otimes\cup} : U^2 \to \mathscr{P}(U)$ *that* returns two *canonical combinations of inputs:*

$$\Phi_{\otimes\cup}\big((H_1,\star_1),(H_2,\star_2)\big) \ := \ \Big\{ \ (H_1 \times H_2,\ \star_\otimes)\ ,\ (H_1 \uplus H_2,\ \star_\uplus) \ \Big\},$$

*where*

$$\star_\otimes\big((x_1,y_1),(x_2,y_2)\big) \ := \ \star_1(x_1,x_2) \ \times \ \star_2(y_1,y_2) \ \subseteq \ H_1 \times H_2$$

(tensor/product *hyperoperation), and* $\star_\uplus$ *is as in Example 2.21.*

Uniform constructors:

$$\Gamma_{\otimes}(H_1,H_2)=H_1\times H_2,\qquad \Pi^{\star}_{\otimes}(\star_1,\star_2)=\star_{\otimes};\quad \Gamma_{\uplus}(H_1,H_2)=H_1\uplus H_2,\quad \Pi^{\star}_{\uplus}(\star_1,\star_2)=\star_{\uplus}.$$

(MH1) Closure: *for all* $(x_i,y_i)\in H_1\times H_2$*,*

$$\star_{\otimes}\big((x_1,y_1),(x_2,y_2)\big)=\star_1(x_1,x_2)\times\star_2(y_1,y_2)\ \subseteq\ H_1\times H_2,$$

*and* $\star_{\uplus}$ *maps into* $\mathscr{P}(H_1\uplus H_2)$ *by blocks, so both outputs are in* $U$*.* (MH2) Naturality: *if* $\alpha_i:(H_i,\star_i)\cong(H_i',\star_i')$*, then* $\alpha_1\times\alpha_2$ *intertwines* $\star_{\otimes}$ *and* $\alpha_1\uplus\alpha_2$ *intertwines* $\star_{\uplus}$*, yielding isomorphisms of both outputs. Therefore* $\big(U,\Phi_{\otimes\cup}\big)$ *is a Meta–HyperStructure with genuinely* set–valued *meta–output.*

**Theorem 2.23** (Reduction to MetaStructure)**.** *Let* $\mathbb{M}=(U_{\mathrm{MS}},(\Psi_\ell)_{\ell\in\Lambda})$ *be a MetaStructure (objects are* structures*, meta*–operations *are deterministic). Define an embedding*

$$\iota_{\mathrm{MS}\to\mathrm{MH}}:\ U_{\mathrm{MS}}\longrightarrow U$$

*by assigning to each* $\Sigma$*–structure* $\mathbf{C}$ *the* $\Sigma$*–hyperstructure*

$$\widehat{\mathbf{C}}=\Big(|\mathbf{C}|,\ (f^{\mathbf{C}}),\ (R^{\mathbf{C}}),\ (f^{\sharp}_{\mathbf{C}})_{f\in\mathsf{Func}}\Big),\qquad f^{\sharp}_{\mathbf{C}}(A_1,\dots,A_m):=\big\{\,f^{\mathbf{C}}(a_1,\dots,a_m)\mid a_i\in A_i\,\big\}.$$

*Define* $\Phi_\ell$ *on* $U$ *by*

$$\Phi_\ell\big(\widehat{\mathbf{C}}_1,\dots,\widehat{\mathbf{C}}_{k_\ell}\big):=\Big\{\ \widehat{\Psi_\ell(\mathbf{C}_1,\dots,\mathbf{C}_{k_\ell})}\ \Big\}.$$

*Then* $\mathbb{MH}=(U,(\Phi_\ell)_{\ell\in\Lambda})$ *is a Meta–HyperStructure and the forgetful functor* $F:U\to U_{\mathrm{MS}}$*,* $F(\widehat{\mathbf{C}})=\mathbf{C}$*, satisfies*

$$F\circ\Phi_\ell\ =\ \Psi_\ell\circ(F,\dots,F)\qquad (\textit{equality of}\ \text{functions}).$$

*Consequently, Meta–HyperStructure strictly generalizes MetaStructure.*

*Proof.* For each $\mathbf{C}$, $\widehat{\mathbf{C}}$ is a $\Sigma$–hyperstructure: $f^{\sharp}_{\mathbf{C}}$ is the standard subset–lifting, so closure holds. Uniformity is clear: the recipes "take carriers/operations/relations as in $\Psi_\ell$ and hyperops as subset–liftings of the resulting operations" do not depend on inputs beyond their interpretations. Hence $\Phi_\ell$ satisfies (MH1) and (MH2). The equality $F\circ\Phi_\ell=\Psi_\ell\circ(F,\dots,F)$ follows by construction (both sides output the same $\Sigma$–structure); thus $\mathbb{MH}$ projects to $\mathbb{M}$, proving generalization. Strictness: take any nontrivial $f^{\sharp}$ (e.g. $m=1$), which has no counterpart in a pure MetaStructure; hence there are Meta–HyperStructures with genuinely hyper–valued internal operations. ☐

**Theorem 2.24** (Reduction to HyperStructure)**.** *Let* $\mathscr{H}=\big(\mathscr{P}(S),\circ\big)$ *be a classical hyperstructure on a nonempty set* $S$ *(with one fixed arity* $m$ *for* $\circ$*). Consider the* purely relational *signature* $\Sigma_0$ *(no function symbols, no relations), and set*

$$U:=\Big\{\ \mathbf{X}_A:=\big(A;\ \textit{no } f \textit{ or } R;\ \circ_A\big)\ \Big|\ A\subseteq S\ \Big\},$$

*where* $\circ_A$ *is the restriction of* $\circ$ *to* $\mathscr{P}(A)^m\to\mathscr{P}(A)$ *(i.e.* $\circ_A(\vec{B}):=\circ(\vec{B})\cap A$ *for* $\vec{B}\in\mathscr{P}(A)^m$*). Define a single meta–hyperoperation* $\Phi:U^m\to\mathscr{P}(U)$ *by*

$$\Phi\big(\mathbf{X}_{A_1},\dots,\mathbf{X}_{A_m}\big)\ :=\ \Big\{\ \mathbf{X}_C\ \Big|\ C\in\circ(A_1,\dots,A_m)\ \Big\}.$$

*Then* $\mathbb{MH}=(U,\{\Phi\})$ *is a Meta–HyperStructure and, for the forgetful map* $F:U\to\mathscr{P}(S)$*,* $F(\mathbf{X}_A)=A$*, one has the exact identity of hyperoperations*

$$F\big(\Phi(\mathbf{X}_{A_1},\dots,\mathbf{X}_{A_m})\big)\ =\ \circ(A_1,\dots,A_m).$$

*Therefore Meta–HyperStructure strictly generalizes HyperStructure.*

*Proof.* (MH1) Closure is immediate: $C\in\circ(A_1,\dots,A_m)\subseteq S$ implies $\mathbf{X}_C\in U$. (MH2) Natural/iso–invariance is trivial since $\Sigma_0$ has no symbols and isomorphisms of pure carriers are bijections; $\Phi$ is defined purely in terms of $\circ$ on carriers. For the identity, by definition

$$F\big(\Phi(\mathbf{X}_{A_1},\dots,\mathbf{X}_{A_m})\big)=\big\{\,C\mid C\in\circ(A_1,\dots,A_m)\,\big\}=\circ(A_1,\dots,A_m).$$

Strictness holds because the meta–level outputs a *set* of hyperstructures; in a plain hyperstructure there is no second level of hyper–valued meta–combination of *objects*. ☐

### 2.4. iterated Meta-HyperStructure(HyperStructure of HyperStructure of ... of HyperStructure)

An iterated Meta-HyperStructure recursively layers hyperstructures; elements themselves are hyperstructures. Meta-hyperoperations uniformly combine levels, ensuring closure, isomorphism-invariance, and hierarchical, multi-level, set-valued combinations across structural depths.

**Definition 2.25** (Level ladder for hyperstructures)**.** *Fix a nonempty set $H$ of atoms. Define inductively*

$$\mathsf{E}^0(H) := H, \qquad \mathsf{E}^{t+1}(H) := \left\{ \mathbf{H} \in \mathrm{HypStr}_\Sigma \;\middle|\; |\mathbf{H}| \subseteq \mathsf{E}^t(H) \right\}.$$

**Definition 2.26** (Iterated Meta–HyperStructure of depth $t$)**.** *An* Iterated Meta–HyperStructure of depth $t$ *is a pair*

$$\mathbb{MH}^{(t)} = \left( U^{(t)},\, (\Phi_\ell^{(t)})_{\ell\in\Lambda} \right)$$

*with $U^{(t)} \subseteq \mathsf{E}^t(H)$ nonempty and meta–hyperoperations*

$$\Phi_\ell^{(t)} :\; \left(U^{(t)}\right)^{k_\ell} \longrightarrow \mathscr{P}\left(U^{(t)}\right)$$

*constructed by* level–$t$ *versions of the uniform recipes $\Gamma_\ell, \Lambda_\ell^f, \Xi_\ell^R, \Pi_\ell^q$ (as in Definition 2.19) so that all outputs lie again in $\mathsf{E}^t(H)$ and (MH1)–(MH2) hold at level $t$.*

**Example 2.27** (Depth 2 over hypermagmas: "closure by product $\otimes$ and disjoint union $\uplus$")**.** *Let $\Sigma_{\mathrm{hm}}$ be the single–sorted signature with one binary hyperoperation symbol $\star$. A level–1 object is a hypermagma $\mathbf{M} = (M, \star_{\mathbf{M}})$ with $\star_{\mathbf{M}} : M \times M \to \mathscr{P}(M)$; thus $\mathsf{E}^1(H)$ is the class of all such $\mathbf{M}$ with $M \subseteq H$.*
*Define two* level–1 constructors *on hypermagmas:*

$$\mathbf{M}_1 \otimes \mathbf{M}_2 := \left(M_1 \times M_2,\, \star_\otimes\right), \quad \star_\otimes\left((x_1,y_1),(x_2,y_2)\right) := \star_{\mathbf{M}_1}(x_1,x_2) \times \star_{\mathbf{M}_2}(y_1,y_2),$$

$$\mathbf{M}_1 \uplus \mathbf{M}_2 := \left(M_1 \uplus M_2,\, \star_\uplus\right), \quad \star_\uplus\left((i,x),(j,y)\right) := \begin{cases} \{1\} \times \star_{\mathbf{M}_1}(x,y), & (i,j) = (1,1), \\ \{2\} \times \star_{\mathbf{M}_2}(x,y), & (i,j) = (2,2), \\ \varnothing, & i \neq j. \end{cases}$$

*A* level–2 object *is a pair $\mathbf{X} = (X, \circ_{\mathbf{X}})$ with $X \subseteq \mathsf{E}^1(H)$ and a binary hyperoperation $\circ_{\mathbf{X}} : \mathscr{P}(X) \times \mathscr{P}(X) \to \mathscr{P}(X)$ defined by*

$$A \circ_{\mathbf{X}} B \;:=\; \left\{ \mathbf{M}_1 \otimes \mathbf{M}_2 \;\middle|\; \mathbf{M}_1 \in A,\, \mathbf{M}_2 \in B \right\} \cup \left\{ \mathbf{M}_1 \uplus \mathbf{M}_2 \;\middle|\; \mathbf{M}_1 \in A,\, \mathbf{M}_2 \in B \right\},$$

*and we* require *$X$ to be closed under $\otimes$ and $\uplus$ so that $A \circ_{\mathbf{X}} B \subseteq X$. Let $U^{(2)}$ be the class of all such $\mathbf{X}$ (objects in $\mathsf{E}^2(H)$).*
*Define the binary meta–hyperoperation $\Phi^{(2)}_{\mathrm{merge}} : (U^{(2)})^2 \to \mathscr{P}(U^{(2)})$ by*

$$\Phi^{(2)}_{\mathrm{merge}}(\mathbf{X}_1, \mathbf{X}_2) \;:=\; \left\{ \mathbf{X} \right\}, \quad \text{where } X := \mathrm{Cl}_{\otimes,\uplus}\left(X_1 \cup X_2\right),\; \circ_{\mathbf{X}} \text{ as above.}$$

Closure (MH1): *by construction $X$ is closed under $\otimes, \uplus$, hence $\mathbf{X} \in U^{(2)}$.* Naturality (MH2): *isomorphisms of level–*1 *hypermagmas extend to blockwise/product isomorphisms of $\otimes, \uplus$; therefore level–2 bijections transport $\circ_{\mathbf{X}}$ componentwise. Thus $\left(U^{(2)}, \Phi^{(2)}_{\mathrm{merge}}\right)$ is an Iterated Meta–HyperStructure of depth* 2.

**Example 2.28** (Arbitrary depth $t \geq 2$: unit–lifted product vs. union (set–valued meta–output))**.** *Retain $\Sigma_{\mathrm{hm}}$ and the level–*1 *constructors $\otimes, \uplus$. For any $t \geq 2$, define $U^{(t)} \subseteq \mathsf{E}^t(H)$ to be the class of pairs $\mathbf{X}^{(t)} = \left(X^{(t)}, \circ^{(t)}_{\mathbf{X}}\right)$ with $X^{(t)} \subseteq \mathsf{E}^{t-1}(H)$ such that $X^{(t)}$ is closed under $\otimes, \uplus$ (applied to its* elements*, which are level–$(t-1)$ hypermagmas), and*

$$A \circ^{(t)}_{\mathbf{X}} B \;:=\; \left\{ \mathbf{M} \otimes \mathbf{N} \;\middle|\; \mathbf{M} \in A,\, \mathbf{N} \in B \right\} \;\subseteq\; X^{(t)}.$$

*(Using only $\otimes$ keeps the formula minimal; a symmetric variant uses $\uplus$.)*
*Define a binary meta–hyperoperation*

$$\Phi^{(t)}_{\otimes\cup} : (U^{(t)})^2 \longrightarrow \mathscr{P}\left(U^{(t)}\right)$$

*by returning* two *canonical level–$t$ outputs:*

$$\Phi^{(t)}_{\otimes\cup}\left(\mathbf{X}^{(t)}_1, \mathbf{X}^{(t)}_2\right) := \Big\{ \underbrace{\left( \mathrm{Cl}_\otimes(X^{(t)}_1 \cup X^{(t)}_2),\; \circ^{(t)}_\otimes \right)}_{\textit{product–closure object}},\; \underbrace{\left( \mathrm{Cl}_\uplus(X^{(t)}_1 \cup X^{(t)}_2),\; \circ^{(t)}_\uplus \right)}_{\textit{union–closure object}} \Big\},$$

*with internal hyperoperations given (uniformly) by*

$$A \circ^{(t)}_\otimes B := \{\mathbf{M} \otimes \mathbf{N} \mid \mathbf{M} \in A, \mathbf{N} \in B\}, \qquad A \circ^{(t)}_\uplus B := \{\mathbf{M} \uplus \mathbf{N} \mid \mathbf{M} \in A, \mathbf{N} \in B\}.$$

Closure (MH1): *each displayed carrier is closed by definition of* Cl*, and the hyperoperations map $\mathscr{P}(X^{(t)})^2$ into $\mathscr{P}(X^{(t)})$.* Naturality (MH2): *any isomorphisms between level–$(t-1)$ elements transport $\otimes, \uplus$ to isomorphic outputs; hence the induced bijections between carriers commute with $\circ^{(t)}_\otimes, \circ^{(t)}_\uplus$. Therefore $\left(U^{(t)}, \Phi^{(t)}_{\otimes\cup}\right)$ is an Iterated Meta–HyperStructure of depth $t$ with genuinely* set–valued *meta–output.*

**Theorem 2.29** (Iteration strictly generalizes)**.** *For $t > s \geq 1$, there is a fully faithful embedding (level–raising)*

$$\iota_{s\to t} : \mathbb{MH}^{(s)} \hookrightarrow \mathbb{MH}^{(t)}$$

*defined on objects by the $r := (t-s)$–fold* unit lift

$$\mathsf{U}^r : \mathsf{E}^s(H) \longrightarrow \mathsf{E}^t(H), \qquad \mathbf{X} \mapsto \underbrace{\mathsf{U}(\cdots\mathsf{U}(\mathbf{X})\cdots)}_{r \text{ times}},$$

*where* $\mathsf{U}$ *turns an object into the singleton hyperstructure on it, and on meta–hyperoperations by*

$$\Phi_\ell^{(s)} \mapsto (\Phi_\ell^{(s)})^{\uparrow} : (\mathsf{U}^r(\mathbf{X}_1),\ldots,\mathsf{U}^r(\mathbf{X}_{k_\ell})) \longmapsto \mathsf{U}^r\big(\Phi_\ell^{(s)}(\mathbf{X}_1,\ldots,\mathbf{X}_{k_\ell})\big).$$

*Hence Iterated Meta–HyperStructure strictly generalizes Meta–HyperStructure (the $t$=1 case).*

*Proof.* Well–definedness: $\mathsf{U}$ maps $\mathsf{E}^u$ to $\mathsf{E}^{u+1}$ by wrapping any object into a singleton carrier and lifting all internal operations, relations, and hyperoperations along that inclusion; iterating $r$ times lands in $\mathsf{E}^t$. Closure and naturality are preserved by functoriality of the recipes and the fact that $\mathsf{U}$ is natural with respect to isomorphisms. Fullness/faithfulness follow since any morphism between lifted singletons descends uniquely to a morphism between their *contents*. Strictness: there exist level–$t$ objects whose carriers contain two nonisomorphic level–$(t-1)$ hyperstructures; they cannot lie at level $s < t$ by Definition 2.25. □

**Theorem 2.30** (Iterated Meta–HyperStructure subsumes Iterated MetaStructure)**.** *Let $\mathbb{M}^{(t)}$ be an Iterated MetaStructure of depth $t$ (objects are $\Sigma$–structures at level $t$ with deterministic meta–operations). Define an embedding*

$$\iota : \mathbb{M}^{(t)} \hookrightarrow \mathbb{MH}^{(t)}$$

*by sending each level–$t$ structure $\mathbf{C}$ to*

$$\widehat{\mathbf{C}} = \Big(|\mathbf{C}|,\ (f^{\mathbf{C}}),\ (R^{\mathbf{C}}),\ (f^{\sharp}_{\mathbf{C}})_{f\in\mathsf{Func}}\Big), \quad f^{\sharp}_{\mathbf{C}}(A_1,\ldots,A_m) = \{f^{\mathbf{C}}(a_1,\ldots,a_m) \mid a_i \in A_i\}$$

*(all at level $t$), and by lifting meta–operations as singletons:*

$$\Phi_\ell^{(t)}(\widehat{\mathbf{C}}_1,\ldots,\widehat{\mathbf{C}}_{k_\ell}) := \{\ \Psi_\ell^{(t)}\widehat{(\mathbf{C}_1,\ldots,\mathbf{C}_{k_\ell})}\ \}.$$

*Then the forgetful functor $F$ from $\mathbb{MH}^{(t)}$ to the underlying iterated MetaStructure satisfies $F\circ\iota = \mathrm{id}$ and $F\circ\Phi_\ell^{(t)} = \Psi_\ell^{(t)}\circ(F,\ldots,F)$. Hence Iterated Meta–HyperStructure generalizes Iterated MetaStructure.*

*Proof.* Exactly as in Theorem 2.23, now performed levelwise at depth $t$. □

### 2.5. Meta-SuperHyperStructure(SuperHyperStructure of SuperHyperStructure)

Let $H$ be a fixed nonempty *ground set* (atoms). For any set $X$ and any $r \in \mathbb{N}_0$, define the iterated powerset

$$\mathscr{P}^0(X) := X, \qquad \mathscr{P}^{r+1}(X) := \mathscr{P}\big(\mathscr{P}^r(X)\big),$$

and the *iterated singleton embedding* $\delta_X^{(r)} : X \to \mathscr{P}^r(X)$ by

$$\delta_X^{(0)}(x) := x, \qquad \delta_X^{(r+1)}(x) := \big\{\delta_X^{(r)}(x)\big\}.$$

When the ground set is clear we write simply $\delta^{(r)}$. For $r \geq 1$ we also use the canonical extension

$$(\delta^{(r)})^{[m]} : \big(\mathscr{P}^m(X)\big)^k \longrightarrow \big(\mathscr{P}^{m+r}(X)\big)^k, \qquad (\delta^{(r)})^{[m]}(A_1,\ldots,A_k) := \big(\delta^{(r)}_{\mathscr{P}^m(X)}(A_1),\ldots,\delta^{(r)}_{\mathscr{P}^m(X)}(A_k)\big).$$

Fix a *profile* (internal type)

$$\Theta = (m_0, n_0, k_0) \in \mathbb{N}_0 \times \mathbb{N}_0 \times \mathbb{N}_{\geq 1}.$$

A $\Theta$–*superhyperoperation on* $X$ is a map

$$\star : \big(\mathscr{P}^{m_0}(X)\big)^{k_0} \longrightarrow \mathscr{P}^{n_0}(X).$$

**Definition 2.31** ($\Theta$–SuperHyperStructure over $X$)**.** *A* $\Theta$–SuperHyperStructure over a base set $X$ *is a pair*

$$\mathbf{S} = (\ \mathscr{P}^{m_0}(X),\ \star_{\mathbf{S}}\ ), \qquad \star_{\mathbf{S}} : \big(\mathscr{P}^{m_0}(X)\big)^{k_0} \to \mathscr{P}^{n_0}(X).$$

*We write* $\mathrm{SH}_\Theta(X)$ *for the class of all such* $\mathbf{S}$ *(up to isomorphism induced by bijections $X \to X$).*

**Remark 2.32.** *Classical hyperstructures of the form $(\mathscr{P}(X),\circ)$ with a $k$–ary hyperoperation $\circ : \mathscr{P}(X)^k \to \mathscr{P}(X)$ correspond to the profile* $\Theta = (1,1,k)$.

Intuitively, objects are *superhyperstructures* on the *same base set* $H$, and meta–*superhyperoperations* send tuples of such objects to *(iterated) sets* of such objects; internally, each output object is built from the input objects by a *uniform constructor* acting on the internal $\Theta$–superhyperoperations.

**Definition 2.33** (Meta–SuperHyperStructure of profile $\Theta$ over $H$)**.** *A* Meta–SuperHyperStructure *(MSH) of profile $\Theta$ over $H$ is a pair*

$$\mathbb{MSH} = \Big( U \subseteq \mathrm{SH}_\Theta(H), \ \ (\Phi_\ell)_{\ell \in \Lambda} \Big)$$

*such that, for each label $\ell \in \Lambda$,*

- *the* meta–arity $s_\ell \in \mathbb{N}$ *and the* meta–height $r_\ell \in \mathbb{N}_0$ *are fixed;*
- *the* meta–superhyperoperation

$$\Phi_\ell : \ U^{s_\ell} \longrightarrow \mathscr{P}^{r_\ell}(U)$$

*is given by a* uniform constructor $\Pi_\ell$ *as follows: for any inputs $\mathbf{S}_1, \dots, \mathbf{S}_{s_\ell} \in U$ with internal operations $\star_{\mathbf{S}_i} : \big(\mathscr{P}^{m_0}(H)\big)^{k_0} \to \mathscr{P}^{n_0}(H)$, every output $\mathbf{T} \in \Phi_\ell(\mathbf{S}_1, \dots, \mathbf{S}_{s_\ell})$ satisfies*

$$\star_{\mathbf{T}} \ = \ \Pi_\ell\big(\star_{\mathbf{S}_1}, \dots, \star_{\mathbf{S}_{s_\ell}}\big), \tag{2}$$

*where $\Pi_\ell$ is a fixed operation*

$$\Pi_\ell : \ \Big[ \big(\mathscr{P}^{m_0}(H)\big)^{k_0} \to \mathscr{P}^{n_0}(H) \Big]^{s_\ell} \longrightarrow \Big(\mathscr{P}^{m_0}(H)\Big)^{k_0} \to \mathscr{P}^{n_0}(H).$$

***Axioms.***

***(MSH1) Closure:*** $\Phi_\ell(\mathbf{S}_1, \dots, \mathbf{S}_{s_\ell}) \subseteq \mathscr{P}^{r_\ell}(U)$.

***(MSH2) Naturality:*** *for any bijection $\alpha : H \to H$, let $\alpha^{[m_0]}, \alpha^{[n_0]}$ act on $\mathscr{P}^{m_0}(H), \mathscr{P}^{n_0}(H)$ by image. If $\mathbf{S}_i'$ is obtained from $\mathbf{S}_i$ by transporting $\star_{\mathbf{S}_i}$ via $\alpha$, then every $\mathbf{T} \in \Phi_\ell(\mathbf{S}_1, \dots, \mathbf{S}_{s_\ell})$ transports to some $\mathbf{T}' \in \Phi_\ell(\mathbf{S}_1', \dots, \mathbf{S}_{s_\ell}')$ with $\star_{\mathbf{T}'} = \alpha^{[n_0]} \circ \star_{\mathbf{T}} \circ (\alpha^{[m_0]})^{-1}$.*

**Remark 2.34** (Concrete examples of $\Pi_\ell$)**.** *Typical uniform constructors include the pointwise convex (or union/intersection) combination*

$$\big(\Pi_\ell(\star_1, \star_2)\big)(\vec{A}) \ := \ \star_1(\vec{A}) \ \cup \ \star_2(\vec{A}), \qquad \big(\Pi_\ell(\star_1, \star_2)\big)(\vec{A}) \ := \ \star_1(\vec{A}) \ \cap \ \star_2(\vec{A}),$$

*and the* convolution *constructor*

$$\big(\Pi_\ell(\star_1, \star_2)\big)(\vec{A}) \ := \ \bigcup_{\substack{\vec{B}, \vec{C} \\ \vec{B} \cup \vec{C} = \vec{A}}} \ \star_1(\vec{B}) \ \cup \ \star_2(\vec{C}),$$

*all of which clearly satisfy (MSH2).*

**Example 2.35** (Pointwise union on binary superhyperoperations)**.** *Fix a nonempty base set $H$ and the profile $\Theta = (m_0, n_0, k_0) = (1, 1, 2)$, so each object $\mathbf{S} \in \mathrm{SH}_\Theta(H)$ has an internal binary superhyperoperation $\star_{\mathbf{S}} : \mathscr{P}(H) \times \mathscr{P}(H) \to \mathscr{P}(H)$. Let $U := \mathrm{SH}_\Theta(H)$.*
*Define a binary meta–superhyperoperation $\Phi_\cup : U^2 \to U$ (meta–arity $s_\ell = 2$, meta–height $r_\ell = 0$) by the* uniform constructor

$$\Pi_\cup(\star_1, \star_2)(A, B) \ := \ \star_1(A, B) \ \cup \ \star_2(A, B) \qquad (A, B \subseteq H).$$

*Set $\Phi_\cup(\mathbf{S}_1, \mathbf{S}_2) := (\mathscr{P}(H), \Pi_\cup(\star_{\mathbf{S}_1}, \star_{\mathbf{S}_2}))$.*
(MSH1) Closure: $\Pi_\cup(\star_1, \star_2)$ *maps $\mathscr{P}(H)^2$ into $\mathscr{P}(H)$, hence the output lies in $U$.* (MSH2) Naturality: *for any bijection $\alpha : H \to H$,*

$$\alpha^{[1]}\big(\Pi_\cup(\star_1, \star_2)(A, B)\big) = \alpha^{[1]}\big(\star_1(A, B) \cup \star_2(A, B)\big) = (\alpha^{[1]} \circ \star_1)(A, B) \ \cup \ (\alpha^{[1]} \circ \star_2)(A, B),$$

*so $\alpha$ transports $\Pi_\cup(\star_1, \star_2)$ to $\Pi_\cup(\alpha^{[1]} \circ \star_1 \circ (\alpha^{[1]})^{-1}, \, \alpha^{[1]} \circ \star_2 \circ (\alpha^{[1]})^{-1})$, as required.*
Concrete check: *let $H = \{0, 1\}$, $\star_\cup(A, B) = A \cup B$ and $\star_\cap(A, B) = A \cap B$. Then for $A = \{0\}, B = \{1\}$,*

$$\Pi_\cup(\star_\cup, \star_\cap)(A, B) = (A \cup B) \cup (A \cap B) = \{0, 1\} \cup \varnothing = \{0, 1\} \in \mathscr{P}(H).$$

*Thus $\big(U, \Phi_\cup\big)$ is a Meta–SuperHyperStructure of profile $\Theta$.*

**Example 2.36** (Kleene-type closure of a superhyperoperation)**.** *Fix the same base $H$ and profile $\Theta = (1, 1, 2)$ and let $U := \mathrm{SH}_\Theta(H)$.*
*Define a* unary *meta–superhyperoperation $\Phi_* : U \to U$ (meta–arity 1, meta–height 0) by the uniform constructor that unions all finite self–compositions of the internal operation:*

$$\Pi_*(\star)\,(A, B) \ := \ \bigcup_{n \geq 1} \star^{\langle n \rangle}(A, B),$$

*where $\star^{\langle 1 \rangle} := \star$ and recursively*

$$\star^{\langle n+1 \rangle}(A, B) \ := \ \star\big(\star^{\langle n \rangle}(A, B), \ \star^{\langle n \rangle}(A, B)\big).$$

*Set $\Phi_*(\mathbf{S}) := (\mathscr{P}(H), \Pi_*(\star_{\mathbf{S}}))$.*
(MSH1) Closure: *every $\star^{\langle n \rangle}(A, B) \subseteq H$; hence their union lies in $\mathscr{P}(H)$.* (MSH2) Naturality: *for any bijection $\alpha : H \to H$, transport respects unions and the recursive definition:*

$$\alpha^{[1]} \circ \Pi_*(\star) \circ (\alpha^{[1]})^{-1} = \Pi_*\Big( \ \alpha^{[1]} \circ \star \circ (\alpha^{[1]})^{-1} \ \Big).$$

*Therefore $\big(U, \Phi_*\big)$ is a Meta–SuperHyperStructure.*
Concrete check: *for $\star_\cup(A, B) = A \cup B$ one has $\star_\cup^{\langle 1 \rangle}(A, B) = A \cup B$ and $\star_\cup^{\langle n+1 \rangle}(A, B) = \star_\cup(A \cup B, A \cup B) = A \cup B$, hence $\Pi_*(\star_\cup)(A, B) = \bigcup_{n \geq 1}(A \cup B) = A \cup B$.*

**Theorem 2.37** (MSH generalizes MH)**.** *Assume each object of a given Meta–HyperStructure $\mathbb{MH} = (U_{\mathrm{MH}}, (\Psi_\ell)_{\ell\in\Lambda})$ is a classical hyperstructure $(\mathscr{P}(H), \circ)$ of fixed arity k (i.e. profile $\Theta = (1,1,k)$). Define*

$$E : U_{\mathrm{MH}} \longrightarrow U \subseteq \mathrm{SH}_{(1,1,k)}(H), \qquad E\big((\mathscr{P}(H), \circ)\big) := \big(\mathscr{P}(H), \star := \circ\big).$$

*Set, for each $\ell$,*

$$\Phi_\ell\big(E(\mathbf{X}_1), \dots, E(\mathbf{X}_{s_\ell})\big) := \big\{ E(\mathbf{Y}) \;\big|\; \mathbf{Y} \in \Psi_\ell(\mathbf{X}_1, \dots, \mathbf{X}_{s_\ell}) \big\} \in \mathscr{P}^{r_\ell}(U),$$

*with the same meta–height $r_\ell$ as in $\mathbb{MH}$. Then $\mathbb{MSH} = (U, (\Phi_\ell))$ is a Meta–SuperHyperStructure of profile $(1,1,k)$ and the forgetful map*

$$F : U \to U_{\mathrm{MH}}, \quad F\big(\mathscr{P}(H), \star\big) := (\mathscr{P}(H), \star)$$

*satisfies the* strict equality of meta–compositions

$$F\Big(\Phi_\ell\big(E(\mathbf{X}_1), \dots, E(\mathbf{X}_{s_\ell})\big)\Big) = \Psi_\ell(\mathbf{X}_1, \dots, \mathbf{X}_{s_\ell}) \qquad \text{in } \mathscr{P}^{r_\ell}(U_{\mathrm{MH}}). \tag{3}$$

*Hence Meta–SuperHyperStructure strictly generalizes Meta–HyperStructure.*

*Proof.* By construction $E(\mathbf{X})$ is a $\Theta$–superhyperstructure with the same underlying hyperoperation; thus $U \subseteq \mathrm{SH}_{(1,1,k)}(H)$. Each $\Phi_\ell$ is defined pointwise from $\Psi_\ell$, so (MSH1) holds. Naturality (MSH2) is inherited because transporting via a bijection $\alpha : H \to H$ acts on both sides by $(\alpha^{[1]}, \alpha^{[1]})$ on domain/codomain; equality (3) is immediate from the definition of $F$ and $\Phi_\ell$. Strictness follows since MSH allows other profiles (e.g. $(m_0, n_0) \neq (1,1)$) and hyper–valued meta–heights $r_\ell > 0$, absent from a purely deterministic meta–operation setting. ☐

**Theorem 2.38** (MSH generalizes SH)**.** *Fix any $\Theta$ and consider the class $\mathrm{SH}_\Theta(H)$. The map*

$$G : \mathrm{SH}_\Theta(H) \longrightarrow \Big\{\text{MSH of profile } \Theta \text{ over } H\Big\}, \qquad G(\mathbf{S}) := \Big( \{\mathbf{S}\}, \text{ no meta–operations} \Big)$$

*is injective, and the forgetful projection $\mathscr{U} : (U, (\Phi_\ell)) \mapsto U$ satisfies $\mathscr{U}(G(\mathbf{S})) = \{\mathbf{S}\}$. Therefore MSH strictly generalizes SuperHyperStructure.*

*Proof.* Trivial: $G(\mathbf{S})$ is an MSH with $U = \{\mathbf{S}\}$, satisfying (MSH1) vacuously and (MSH2) because no $\Phi_\ell$ are present. Injectivity of $G$ follows from equality of the unique element of $U$. ☐

## 2.6. iterated Meta-SuperHyperStructure(SuperHyperStructure of ... of SuperHyperStructure)

We iterate "superhyperstructure of superhyperstructure" by letting the *base set of the next level* be the *universe of objects* from the previous level.

**Definition 2.39** (Level ladder for superhyperstructures)**.** *Define inductively*

$$\mathsf{E}^0_\Theta(H) := H, \qquad \mathsf{E}^{t+1}_\Theta(H) := \mathrm{SH}_\Theta\big(\mathsf{E}^t_\Theta(H)\big) \qquad (t \geq 0).$$

**Definition 2.40** (Iterated Meta–SuperHyperStructure of depth $t$)**.** *An* Iterated Meta–SuperHyperStructure *(IMSH) of depth $t \geq 1$ and profile $\Theta$ is a pair*

$$\mathbb{MSH}^{(t)} = \Big( U^{(t)} \subseteq \mathsf{E}^t_\Theta(H), \; \big(\Phi^{(t)}_\ell : (U^{(t)})^{s_\ell} \to \mathscr{P}^{r_\ell}(U^{(t)})\big)_{\ell\in\Lambda} \Big),$$

*where each $\Phi^{(t)}_\ell$ is given by a uniform constructor $\Pi_\ell$ acting on the internal $\Theta$–superhyperoperations of elements of $U^{(t)}$, exactly as in (2) but with the base set $X = \mathsf{E}^{t-1}_\Theta(H)$.*

**Example 2.41** (Depth $t \geq 2$: pointwise union of internal superhyperoperations)**.** *Fix a profile $\Theta = (m_0, n_0, k_0)$, a base $H \neq \varnothing$, and the ladder $\mathsf{E}^t_\Theta(H)$ of Definition 2.39. Let $U^{(t)} := \mathsf{E}^t_\Theta(H)$ and write $X := \mathsf{E}^{t-1}_\Theta(H)$. For $\mathbf{S}_i \in U^{(t)}$ let $\star_{\mathbf{S}_i} : \big(\mathscr{P}^{m_0}(X)\big)^{k_0} \to \mathscr{P}^{n_0}(X)$ be its internal $\Theta$–superhyperoperation.*
*Define a binary meta–superhyperoperation $\Phi^{(t)}_\cup : (U^{(t)})^2 \to U^{(t)}$ by the uniform constructor*

$$\Pi_\cup(\star_1, \star_2)(\vec{A}) := \star_1(\vec{A}) \cup \star_2(\vec{A}) \qquad \big(\vec{A} \in (\mathscr{P}^{m_0}(X))^{k_0}\big).$$

*Set $\Phi^{(t)}_\cup(\mathbf{S}_1, \mathbf{S}_2) := \big(\mathscr{P}^{m_0}(X), \Pi_\cup(\star_{\mathbf{S}_1}, \star_{\mathbf{S}_2})\big)$.*
Closure (MSH1): $\star_1(\vec{A}), \star_2(\vec{A}) \subseteq \mathscr{P}^{n_0}(X) \Rightarrow \Pi_\cup(\star_1, \star_2)(\vec{A}) \in \mathscr{P}^{n_0}(X)$*; hence the output is in $U^{(t)}$.* Naturality (MSH2): *for any bijection $\alpha : X \to X$,*

$$\alpha^{[n_0]} \circ \Pi_\cup(\star_1, \star_2) \circ (\alpha^{[m_0]})^{-1} = \Pi_\cup\Big( \alpha^{[n_0]} \circ \star_1 \circ (\alpha^{[m_0]})^{-1}, \; \alpha^{[n_0]} \circ \star_2 \circ (\alpha^{[m_0]})^{-1} \Big).$$

Concrete check ($\Theta = (1,1,2)$, $t = 2$): $X = \mathsf{E}^1_\Theta(H) = \mathrm{SH}_\Theta(H)$. *For $A, B \subseteq X$, taking $\star_\cup(A,B) = A \cup B$ and $\star_\cap(A,B) = A \cap B$, one gets $\Pi_\cup(\star_\cup, \star_\cap)(A,B) = (A \cup B) \cup (A \cap B) = A \cup B \in \mathscr{P}(X)$.*

**Example 2.42** (Depth $t \geq 2$: iterated (Kleene-type) closure of internal superhyperoperations)**.** *With the same* $\Theta, H, t$ *and* $X = \mathsf{E}_{\Theta}^{t-1}(H)$ *as above, define a* unary *meta–superhyperoperation* $\Phi_*^{(t)} : U^{(t)} \to U^{(t)}$ *by the uniform constructor*

$$\Pi_*(\star)(\vec{A}) \;:=\; \bigcup_{n\geq 1} \star^{\langle n\rangle}(\vec{A}), \quad \star^{\langle 1\rangle} := \star, \;\; \star^{\langle n+1\rangle}(\vec{A}) := \star\big(\underbrace{\star^{\langle n\rangle}(\vec{A}),\ldots,\star^{\langle n\rangle}(\vec{A})}_{k_0 \text{ slots}}\big).$$

*Set* $\Phi_*^{(t)}(\mathbf{S}) := \big(\mathscr{P}^{m_0}(X),\, \Pi_*(\star_{\mathbf{S}})\big)$.
Closure (MSH1): *each* $\star^{\langle n\rangle}(\vec{A}) \in \mathscr{P}^{n_0}(X)$ *by induction; unions remain in* $\mathscr{P}^{n_0}(X)$*, hence the output belongs to* $U^{(t)}$. Naturality (MSH2): *for any bijection* $\alpha : X \to X$*, transport commutes with composition and union:*

$$\alpha^{[n_0]} \circ \Pi_*(\star) \circ (\alpha^{[m_0]})^{-1} = \Pi_*\Big(\,\alpha^{[n_0]} \circ \star \circ (\alpha^{[m_0]})^{-1}\,\Big).$$

Concrete check ($\Theta = (1,1,2)$, $t = 2$): *if* $\star_{\cup}(A,B) = A \cup B$ *on* $X$*, then* $\star_{\cup}^{\langle n\rangle}(A,B) = A \cup B$ *for all* $n$*, so* $\Pi_*(\star_{\cup})(A,B) = \bigcup_{n\geq 1}(A\cup B) = A\cup B \in \mathscr{P}(X)$.

**Theorem 2.43** (Iteration strictly generalizes)**.** *For* $t > s \geq 1$ *there exists an* injective, meta–compatible *embedding*

$$\iota_{s\to t} :\; \mathbb{MSH}^{(s)} \;\hookrightarrow\; \mathbb{MSH}^{(t)}$$

*defined objectwise by the* constant-$\delta$ lift*: for* $\mathbf{S} \in \mathsf{E}_{\Theta}^{s}(H)$*, put* $\iota_{s\to t}(\mathbf{S}) = \mathbf{S}^{\uparrow} \in \mathsf{E}_{\Theta}^{t}(H)$ *with internal operation*

$$\star_{\mathbf{S}^{\uparrow}}(\vec{A}) \;:=\; \delta^{(n_0-(t-s))}_{\mathsf{E}_{\Theta}^{t-1}(H)}\big(\underbrace{\delta^{(t-s)}_{\mathsf{E}_{\Theta}^{s-1}(H)}(\mathbf{S})}_{\in \mathsf{E}_{\Theta}^{t-1}(H)}\big) \quad \text{for all } \vec{A} \in \big(\mathscr{P}^{m_0}(\mathsf{E}_{\Theta}^{t-1}(H))\big)^{k_0}, \tag{4}$$

*i.e. a constant hyperoperation returning the (iterated) singleton of* $\mathbf{S}$ *at the required level.* † *Meta–operations are lifted by*

$$\Phi_{\ell}^{(t)}\big(\iota_{s\to t}(\vec{\mathbf{S}})\big) \;:=\; \big(\delta^{(0)}\big)^{[0]}\Big(\,\{\,\iota_{s\to t}(\mathbf{T}) \;\big|\; \mathbf{T} \in \Phi_{\ell}^{(s)}(\vec{\mathbf{S}})\,\}\,\Big) \;\in\; \mathscr{P}^{r_\ell}\big(U^{(t)}\big).$$

*Then for every label* $\ell$ *and every tuple* $\vec{\mathbf{S}} \in (U^{(s)})^{s_\ell}$ *one has the exact identity*

$$\iota_{s\to t}\big(\Phi_{\ell}^{(s)}(\vec{\mathbf{S}})\big) \;=\; \Phi_{\ell}^{(t)}\big(\iota_{s\to t}(\vec{\mathbf{S}})\big) \quad \text{as elements of } \mathscr{P}^{r_\ell}\big(U^{(t)}\big). \tag{5}$$

*Hence IMSH strictly generalizes MSH (the case* $t{=}1$*).*

*Proof.* The definition (4) yields a well–typed $\Theta$–superhyperoperation on the higher–level base $\mathsf{E}_{\Theta}^{t-1}(H)$ because its output lies in $\mathscr{P}^{n_0}(\mathsf{E}_{\Theta}^{t-1}(H))$ by construction of $\delta^{(\cdot)}$. Injectivity of $\iota_{s\to t}$ holds since distinct $\mathbf{S}$ produce distinct constants. Equality (5) follows from the pointwise definition of $\Phi_{\ell}^{(t)}$ as the image of $\Phi_{\ell}^{(s)}$ under $\iota_{s\to t}$; (MSH1)–(MSH2) are preserved because constant lifts commute with transport along bijections of the base (they are defined via $\delta^{(\cdot)}$ only). Strictness is immediate: IMSH admits levels $t > 1$ which cannot be realized at level 1 by definition of the ladder $\mathsf{E}_{\Theta}^{t}(H)$. □

**Theorem 2.44** (IMSH subsumes Iterated Meta–HyperStructure)**.** *Let* $\mathbb{MH}^{(t)}$ *be any Iterated Meta–HyperStructure of depth* $t$ *whose objects are (classical) hyperstructures* $\big(\mathscr{P}(\mathsf{E}_{\Theta}^{t-1}(H)), \circ\big)$ *of fixed arity* $k$*. Fix* $\Theta = (1,1,k)$*. The assignment*

$$\mathscr{E} :\; \mathbb{MH}^{(t)} \longrightarrow \mathbb{MSH}^{(t)}, \qquad \big(\mathscr{P}(\mathsf{E}_{\Theta}^{t-1}(H)), \circ\big) \longmapsto \big(\mathscr{P}(\mathsf{E}_{\Theta}^{t-1}(H)), \star := \circ\big),$$

*extended on meta–operations by*

$$\Phi_{\ell}^{(t)}\big(\mathscr{E}(\vec{\mathbf{X}})\big) := \big\{\,\mathscr{E}(\mathbf{Y}) \;\big|\; \mathbf{Y} \in \Psi_{\ell}^{(t)}(\vec{\mathbf{X}})\,\big\},$$

*embeds* $\mathbb{MH}^{(t)}$ *into* $\mathbb{MSH}^{(t)}$ *and obeys the exact identity*

$$F \circ \Phi_{\ell}^{(t)} \circ \mathscr{E}^{\times s_\ell} \;=\; \Psi_{\ell}^{(t)} \quad \text{for all } \ell,$$

*where* $F$ *forgets the "super" label* ($F(\mathscr{P}(X),\star) = (\mathscr{P}(X),\star)$)*. Thus IMSH generalizes Iterated Meta–HyperStructure.*

*Proof.* Identical to Theorem 2.37, carried out at base $X = \mathsf{E}_{\Theta}^{t-1}(H)$. □

## 3. Conclusion

We have introduced and studied the concepts of *MetaStructure*, *Iterated MetaStructure*, *Meta-HyperStructure*, and *Meta-SuperHyperStructure*. These frameworks provide a systematic way of generalizing classical structures, hyperstructures, and superhyperstructures from a meta-level perspective, allowing us to model increasingly complex hierarchies of mathematical objects and their interactions.
In future work, we plan to consider extensions that incorporate uncertainty and multi-valuedness by employing advanced set-theoretic frameworks such as the *Fuzzy Set* [67–70], *Intuitionistic Fuzzy Set* [71–75], *Vague Sets* [76–78], *Hesitant Fuzzy Set* [79–81], *Soft Set* [82–85], *Picture Fuzzy Set* [86–88], *Pythagorean fuzzy set* [89–91], *Neutrosophic Set* [92–97], and *Plithogenic Set* [47, 98–100]. Such extensions may provide a richer and more flexible framework for handling vagueness, indeterminacy, and contradiction in meta-structural systems.

† If $n_0 < (t-s)$, precompose with $(\delta^{(t-s-n_0)})^{[m_0]}$ on inputs and postcompose by $\delta^{(0)}$; in all cases the typing equality in (4) holds by design.

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**Data Availability:** This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

**Ethical Approval:** As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required..

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**Disclaimer**This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

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